Algebraic Geometry With a View Towards Number Theory

From Abstraction to Computation

Robin Truax

December 24, 2021

Contents

Introduction

These notes, compiled from Hartshorne's *Algebraic Geometry*, Vakil's *Foundations of Algebraic Geometry*, and Brian Conrad's course notes from the quarter in which I took Math 216A (transcribed by Vaughan McDonald), are my attempt to organize my own knowledge about algebraic geometry and breath some life into it. I also received help from my friend Sándor Kovács in building geometric intuition. His advice, to work towards being "at a level when you don't even need to translate [between algebra and geometry], because the algebraic and geometric sides live in your brain simultaneously", has shaped my journey into algebraic geometry. These are by far my most ambitious and important notes yet, so I hope you enjoy what you find. As always, let me know at truax [at] stanford [dot] edu if you find any mistakes.

The prerequisites for these notes are: ring theory, field theory, commutative algebra (though important results are proven in the appendix or in my notes on the subject available here), topology (again, important results are proven in the appendix), and mathematical maturity (for example, experience with manifolds may be helpful for intuition about schemes).

Note: All rings are commutative with identity unless otherwise stated.

1 Varieties

Varieties form the basis of classical algebraic geometry, and are useful for building intuition. However, beware of bringing any geometric intuition from this section into other sections without checking it rigorously: varieties are substantially "nicer" than schemes (we will explore the sense in which varieties are literally "nice" schemes in a later section).

1.1 The Zariski Topology

Definition 1.1 (Affine n-Space)**.** Let k be a field. *Affine n-space over k*, denoted $\mathbb{A}_k^n$, is the set of all n-tuples of elements of k; that is, affine n-space over k is the underlying set of the vector space k^n.

Now, let $A = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables over k. Any element $f \in A$ can be interpreted as a function $\mathbb{A}_k^n \to k$ by sending $(a_1, \ldots, a_n)$ to $f(a_1, \ldots, a_n)$ (that is, by sending a point to the element of k given by replacing each variable of f with the respective coordinate of the point).

Definition 1.2 (Zero Set)**.** Suppose $f \in A$ is a polynomial. Then $Z(f) = \{P \mid \mathbb{A}_k^n \mid f(P) = 0\}$ is called the *zero set* of f. More generally, if $S \subseteq A$ is a collection of polynomials, the zero set of S is the collection of points at which every polynomial in S vanishes; that is, $Z(S) = \{P \mid \mathbb{A}_k^n \mid f(P) = 0 \text{ for all } f \in S\}$.

Definition 1.3 (Algebraic Set)**.** A subset Y of $\mathbb{A}_k^n$ is an *(affine) algebraic set* if there exists a subset $S \subseteq A$ such that $Y = Z(S)$.

Proposition 1.4 (Properties of Algebraic Sets)**.** *The union of any finite collection of algebraic sets is an algebraic set. The intersection of any family of algebraic sets is an algebraic set. The empty set and the whole space are algebraic sets.*

Proof. Suppose that $S_1, S_2 \subseteq A$, $Y_1 = Z(S_1)$, and $Y_2 = Z(S_2)$. Then one may easily verify $Y_1 \cup Y_2 = Z(S_1 S_2)$, where $S_1 S_2$ denotes the set of all products of an element of S_1 by an element of S_2. Hence $Y_1 \cup Y_2$ is an algebraic set, so by induction the union of any finite collection of algebraic sets is an algebraic set. On the other hand, given any family $\{S_\lambda\}_{\lambda \in \Lambda}$ of subsets of A and their respective algebraic sets $Y_\lambda = Z(S_\lambda)$, $\bigcap_\lambda Y_\lambda = Z(\bigcup_\lambda S_\lambda)$, so the former is an algebraic set. Finally, $\varnothing = Z(1)$ and $\mathbb{A}_k^n = Z(0)$. □

Definition 1.5 (Zariski Topology)**.** The *Zariski topology* on $\mathbb{A}_k^n$ is the topology given by defining the algebraic sets to be the closed sets. This defines a valid topology by Proposition 1.4.

The following proposition shows that it suffices to consider the zero sets of ideals, rather than arbitrary subsets of A.

Proposition 1.6 (Zero Sets of Ideals). *Suppose that S is a subset of A, and $\langle S\rangle \lhd A$ is the ideal of A generated by S. Then $V(S) = V(\langle S\rangle)$.*

Note 1.7. By Hilbert's Basis Theorem, A is Noetherian. Hence any ideal of A can be generated by finitely many elements. In light of the above proposition, this means that any infinite set S of polynomials in n variables over a field k has a corresponding finite set of polynomials (namely any finite set of generators for $\langle S\rangle$) which vanishes in exactly the same places.

Example 1.8 (The Zariski Topology on $\mathbb{A}^1_k$). In this case, the corresponding ring $A = k[x]$ is a Euclidean domain, hence a principal ideal domain. Therefore, every algebraic set is the set of zeroes of a single polynomial. Now, any finite set appears as the set of zeroes of a single polynomial, and any nonzero polynomial has a finite set of zeroes. Therefore, the closed sets in $\mathbb{A}^1_k$ are all collections of finitely many points (including the empty set, of course), and the entire space.

Definition 1.9 (The Ideal of a Set). Given a set $Y \subseteq \mathbb{A}^n_k$, the *ideal of* Y in A is $I(Y) = \{f \in A \mid f(P) = 0 \text{ for all } P \in Y\}$. It is easy to verify that $I(Y)$ is an ideal, since the sum of any two polynomials vanishing on Y vanishes on Y, and the product of any polynomial with a polynomial vanishing on Y vanishes on Y.

Example 1.10 (Ideals with Equal Algebraic Sets). A reasonable question to ask is if there is a two-way correspondence between ideals and algebraic sets. Certainly, given two different algebraic sets, their ideals will be different (exercise: prove this fact). However, unfortunately, there are different ideals which give the same algebraic set. Consider, for example, (x) and (x^2) in $k[x]$. These ideals are not equal, yet both correspond to the point $(0) \in \mathbb{A}^1_k$. You may complain that this example is cheating: after all, the algebraic set of (f) and (f^n) are equal for any polynomial f and positive integer n. Is there a "less trivial" example, you ask? Indeed, there is not; this is the thrust of Hilbert's Nullstellensatz, which we discuss later.

Proposition 1.11 (More Properties of Algebraic Sets). *Let k be a field and $A = k[x_1, \ldots, x_n]$.*

(1) If $T_1 \subseteq T_2$ are subsets of A, then $Z(T_1) \supseteq T_2$.

(2) Conversely, if $Y_1 \subseteq Y_2$ are subsets of $\mathbb{A}^n_k$, then $I(Y_1) \supseteq I(Y_2)$.

(3) If $Y_1, Y_2 \subseteq \mathbb{A}^n_k$, then $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.

(4) For any subset $Y \subseteq \mathbb{A}^n_k$, $Z(I(Y)) = \overline{Y}$, the closure of Y.

Finally, we have arrived at the first truly nontrivial fact about varieties, known as Hilbert's Nullstellensatz.

Theorem 1.12 (Hilbert's Nullstellensatz). *For any ideal $\mathfrak{a} \lhd A$, $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$, the radical of $\mathfrak{a}$.*

Proof. See Appendix A.2 or my notes on commutative algebra. □

1.2 Affine Varieties

For the rest of this chapter, let k be an ***algebraically closed*** field.

Definition 1.13 (Irreducible Subset). A nonempty topological space X is called *irreducible* if it cannot be expressed as the union $Y_1 \cup Y_2$ of two proper closed subsets $Y_1, Y_2 \subsetneq X$. A nonempty subset Y of a topological space X is called *irreducible* if Y is irreducible when given the subspace topology.

If you have not encountered this definition before (which is understandable, as it does not often appear in the study of "nice" topological spaces), see Appendix B, Results from Topology, for useful facts about irreducible subsets/spaces (in particular equivalent conditions for irreducibility).

Definition 1.14 (Affine and Quasiaffine Varieties). An *(affine algebraic) variety* is an irreducible closed subset of $\mathbb{A}^n_k$; that is, an irreducible algebraic set. An open subset of an affine variety is a *quasi-affine variety.*

Theorem 1.15 (Algebro-Geometric Correspondence). *Suppose that k is an algebraically closed field. Then there is an inclusion-reversing correspondence between types of closed subsets of $\mathbb{A}^n_k$ and types of ideals of $A = k[x_1, \ldots, x_n]$, given by the operations Z and I, as follows:*

1. *Radical ideals of A correspond to closed subsets of $\mathbb{A}^n_k$.*
2. *Prime ideals of A correspond to irreducible closed subsets of $\mathbb{A}^n_k$.*
3. *Maximal ideals of A correspond to single points of $\mathbb{A}^n_k$.*

Proof. The fact that radical ideals of A correspond to closed subsets of $\mathbb{A}^n_k$ follows immediately from Theorem 1.12 and Proposition 1.11. Now, recall that prime ideals are radical; hence for (ii) suffices to show that a radical ideal of A is prime if and only if its corresponding closed subset of $\mathbb{A}^n_k$ is irreducible.

Let $\mathfrak{p} \lhd A$ be prime, and suppose that $Z(\mathfrak{p}) = Y_1 \cup Y_2$. Then $\mathfrak{p} = I(Y_1) \cap I(Y_2)$, so either $\mathfrak{p} = I(Y_1)$ or $\mathfrak{p} = I(Y_2)$. But then either $Y_1 = Z(\mathfrak{p})$ or $Y_2 = Z(\mathfrak{p})$, so $Z(\mathfrak{p})$ is indeed irreducible. On the other hand, suppose that Y is irreducible, and take $fg \in I(Y)$. Then $Y \subseteq Z(fg) = Z(f) \cup Z(g)$, and indeed $Y = (Y \cap Z(f)) \cup (Y \cap Z(g))$ is an expression of Y as the union of two closed subsets of Y. Therefore either $Y \cap Z(f) = Y$ (in which case $Y \subseteq Z(f)$ and $f \in I(Y)$) or $Y \cap Z(g) = Y$ (in which case $Y \subseteq Z(g)$ and $g \in I(Y)$). Hence if $fg \in I(Y)$ then either $f \in I(Y)$ or $g \in I(Y)$, so $I(Y)$ is prime, as desired.

Finally, to prove (iii), it suffices to recognize that the correspondence given by I and Z is inclusion-reversing (see Proposition 1.11), so maximal ideals of A correspond to minimal (nonempty) closed sets of $\mathbb{A}^n_k$. Yet any point of $\mathbb{A}^n_k$ is closed (for example, the point $(a_1, \ldots, a_n)$ is the zero set of the collection $\{x_1 - a_1, \ldots, x_n - a_n\}$) and is clearly minimal. Hence the result follows. ☐

Corollary 1.15.1 (Maximal Ideals in A)**.** *Any maximal ideal of $A = k[x_1, \ldots, x_n]$ has the form $(x_1 - a_1, \ldots, x_n - a_n)$ for $a_1, \ldots, a_n \in k$.*

Proof. Recall from the above theorem that any maximal ideal of A is the ideal corresponding to a point $(a_1, \ldots, a_n) \in \mathbb{A}^n_k$. Yet the ideal corresponding to this point is $(x_1 - a_1, \ldots, x_n - a_n)$. ☐

Corollary 1.15.2. *$\mathbb{A}^n_k$ is irreducible.*

Proof. This follows immediately from $\mathbb{A}^n_k = Z(0)$, since 0 is prime in $k[x_1, \ldots, x_n]$. ☐

Definition 1.16 (Coordinate Ring)**.** Let Y be an affine algebraic set in $\mathbb{A}^k_n$. Then the *coordinate ring of Y*, denoted $A(Y)$ is $k[x_1, \ldots, x_n]/I(Y)$. When Y is a variety, $A(Y)$ is a domain.

The coordinate ring can be considered as the ring of polynomial functions on Y. To see why, notice that the polynomials f and g are equal in $A(Y)$ if and only if f and g differ by a polynomial which vanishes on Y; that is, if and only if f and g give the same outputs on each point of Y. For example, $A(\varnothing)$ is the zero ring (since there is just one function and hence one polynomial function on the empty set), and $A(\mathbb{A}^k_n) = k[x_1, \ldots, x_n]$.

Definition 1.17 (Noetherian Topological Space)**.** A topological space X is called *Noetherian* if it satisfies the descending chain condition for closed subsets: for any descending chain $Y_1 \supseteq Y_2 \supseteq \cdots$ of closed subsets, there is an integer r such that $Y_r = Y_{r+1} = \cdots$ (that is, the chain stabilizes).

Proposition 1.18 (Affine n-Space is Noetherian)**.** *$\mathbb{A}^n_k$ (with the Zariski toplogy) is Noetherian.*

Proof. A descending chain of closed sets in $\mathbb{A}^k_n$ corresponds to an ascending chain of (radical) ideals in $k[x_1, \ldots, x_n]$. Since $k[x_1, \ldots, x_n]$ is Noetherian, the ascending chain of ideals stabilizes, so the descending chain of closed sets stabilizes. ☐

Lemma 1.19 (Irreducible Sets in Unions)**.** *If an irreducible set Z is in the union $X_1 \cup \cdots \cup X_r$ of some irreducible closed sets $X_1, \ldots, X_r$, then $Z \subseteq X_j$ for some j.*

Proof. In this case, $X_i \cap Z$ is a closed set for each i. In particular, $Z = (X_1 \cap Z) \cup \cdots \cup (X_r \cap Z)$, so since Z is irreducible, $Z = X_i \cap Z$ for some i, implying that $Z \subseteq X_i$, as desired. ☐

Proposition 1.20 (Unique Decomposition in Noetherian Spaces)**.** *In a Noetherian topological space X, every nonempty closed subset Y can be expressed as a finite union $Y = Y_1 \cup \cdots \cup Y_r$ of irreducible closed subsets Y_i. If we require that $Y_i \not\supseteq Y_j$ for $i \neq j$, then the Y_i are uniquely determined. They are called the* irreducible components *of Y.*

Proof. Let $\mathscr{S}$ be the set of subsets of X that cannot be written as the union of irreducible subsets. If $\mathscr{S} = \varnothing$, we are done, so assume it is nonempty. Since X is Noetherian, $\mathscr{S}$ has a minimal element $Y \in \mathscr{S}$. Yet Y cannot be irreducible (else it is the union of irreducible subsets), so we can write $Y' \cup Y'' = Y$. By the minimality of Y, both Y' and Y'' are not in $\mathscr{S}$, so they can be written as the union of irreducible subsets. Yet then Y is the union of these two unions, a contradiction.

Now assume the extra condition and suppose there are two such decompositions $Y = Y_1 \cup \cdots \cup Y_r = Y'_{1'} \cup \cdots \cup Y'_{r'}$. Then, for any i, $Y_i \in Y'_{1'} \cup \cdots \cup Y'_{r'}$, so by the above Lemma, $Y_i \subseteq Y'_j$ for some j. By identical logic, $Y'_j \subseteq Y_k$ for some k. Thus $Y_i \subseteq Y'_j \subseteq Y_k$, so by the extra condition $Y_i = Y'_j = Y_k$. By repeating this argument, we see that the irreducible sets in the first decomposition are identical, up to some permutation, to the irreducible sets in the second decomposition. □

Corollary 1.20.1 (Unique Decomposition of Algebraic Sets)**.** *Every algebraic set in $\mathbb{A}^n_k$ can be expressed uniquely as a union of varieties (as long as no variety is allowed to contain another).*

1.3 Dimension

Definition 1.21 (Dimension of a Topological Space)**.** Suppose that X is a topological space. Then the *dimension* of X, denoted $\dim X$, is the supremum of all integers n such that there exists a strict chain $Z_0 \subsetneq Z_1 \subsetneq Z_2 \subsetneq \cdots \subsetneq Z_n$ of closed irreducible subsets of X.

Now, it is easy to demonstrate that $\mathbb{A}^1_k$ has dimension 1, as the only irreducible subsets of $\mathbb{A}^1_k$ are single points and the entire space. One might expect that $\mathbb{A}^n_k$ has dimension n, and indeed this is correct, but it requires some commutative algebra to prove.

First, we will translate the problem into commutative algebra.

Definition 1.22 (Height and Krull Dimension)**.** Suppose that A is a ring. Then the *height* of a prime ideal $\mathfrak{p}$, denoted $\operatorname{ht}\mathfrak{p}$, is the supremum of all integers n such that $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}$ of prime ideals of A culminating in $\mathfrak{p}$. The *Krull dimension* of A, denoted $\dim A$, is the supremum of the heights of all of the prime ideals of A. In particular, the Krull dimension of a field is 0, and the Krull dimension of a PID is 1.

Proposition 1.23 (Dimension of Algebraic Set is Dimension of Ring)**.** *The topological dimension of an affine algebraic set Y is equal to the Krull dimension of the coordinate ring $A(Y)$. In particular, the topological dimension of $\mathbb{A}^n_k$ is equal to the Krull dimension of $A = k[x_1, \ldots, x_n]$.*

Proof. Follows immediately from definitions and Theorem 1.15. □

Now, to show $\dim \mathbb{A}^n_k = n$, it suffices to show that $\dim k[x_1, \ldots, x_n] = n$. This fact feels like it should have a trivial proof, but it does not. Nonetheless, it is true, and a consequence of the following theorem, which we cite in the appendix.

Theorem 1.24. *Let k be a field, and A an integral domain which is a finitely-generated k-algebra. Then the dimension of A is equal to the transcendence degree of the quotient field* $\operatorname{Frac} A$ *over k, and for any prime ideal $\mathfrak{p} \lhd A$, we have* $\operatorname{ht}\mathfrak{p} + \dim A/\mathfrak{p} = \dim A$.

Proof. See Appendix A.3. □

Corollary 1.24.1. *The dimension of $\mathbb{A}^n_k$ is n.*

Proposition 1.25. *If Y is a quasi-affine variety, then* $\dim Y = \dim \overline{Y}$.

Proof. By Proposition 6.23, $\dim Y \le \dim \overline{Y}$. Thus $\dim Y$ is finite, so there is a chain $Z_0 \subseteq \cdots \subseteq Z_n$ of distinct closed irreducible subsets of Y with maximal length. In that case, Z_0 must be a point P, and the chain $P = \overline{Z}_0 \subseteq \cdots \subseteq \overline{Z}_n$ of distinct irreducible closed subsets of $\overline{Y}$ (see the proof of Proposition 6.23) is maximal; that is, it cannot be extended further (see Proposition 6.13). Now, P corresponds to a maximal ideal $\mathfrak{m}$ of the affine coordinate ring $A(\overline{Y})$. The $\overline{Z}_i$ correspond to prime ideals contained in $\mathfrak{m}$, so $\operatorname{ht}\mathfrak{m} = n$. On the other hand, $A(\overline{Y})/\mathfrak{m} \simeq k$. Hence by Theorem 1.24, $n = \dim A(\overline{Y}) = \dim \overline{Y}$. Thus $\dim Y = \dim \overline{Y}$. □

Theorem 1.26 (Krull's Haupidealsatz)**.** *Let A be a Noetherian ring, and let $a \in A$ be a nonunit non-zero divisor. Then every minimal prime ideal containing a has height 1.*

Proof. See Appendix A.3. □

Theorem 1.27. *A Noetherian domain A is a UFD if and only if every prime ideal of height 1 is principal.*

Proof. See Appendix A.3. □

Proposition 1.28. *A variety Y in $\mathbb{A}_k^n$ has dimension $n-1$ if and only if it is the zero set $Z(f)$ of a single nonconstant irreducible polynomial in $A = k[x_1, \ldots, x_n]$.*

Proof. Firstly, because A is a UFD, $Z(f)$ is a variety iff (f) is a prime ideal iff f is irreducible. Now, by Theorem 1.26, $\mathfrak{p}$ has height 1, so $Z(f)$ has dimension $n-1$ by Theorem 1.24. Conversely, a variety of dimension $n-1$ corresponds to a prime ideal $\mathfrak{p}$ of height 1. Since A is a UFD, $\mathfrak{p}$ is principal (see Theorem 1.27), generated by an irreducible polynomial f. Hence $Y = Z(f)$, and we are done. □

1.4 Projective Varieties

For the rest of this section, when it is not specified, k is an algebraically closed field.

Definition 1.29 (Projective n-Space)**.** Let k be a field. Then, *projective n-space over k*, denoted $\mathbb{P}_k^n$, is the set of equivalent classes of nonzero $(n+1)$-tuples $(a_0, \ldots, a_n)$ of elements of k under the equivalence relation given by $(a_0, \ldots, a_n) \sim (\lambda a_0, \ldots, \lambda a_n)$ for all $\lambda \in k^\times$. A representative $(a_0, \ldots, a_n)$ of such an equivalence class P is called a *set of homogeneous coordinates for the point P*, and denoted $(a_0 : a_1 : \cdots : a_n)$.

One may visualize $\mathbb{P}_k^n$ as the set of lines through the origin in $(n+1)$-affine space. One may decompose projective space as $\mathbb{A}_k^n \cup \mathbb{P}_k^{n-1}$ by consdering the plane $a_0 = 1$. This allows for an inductive decomposition of projective n-space as a disjoint union $\mathbb{A}_k^n \sqcup \mathbb{A}_k^{n-1} \sqcup \cdots \sqcup \mathbb{A}_k^1 \sqcup \mathbb{A}_k^0$. Covering projective space with affine spaces is a special case of a general technique which generalizes, in the case of defining schemes.

Definition 1.30 (Graded Ring)**.** A *graded ring* is a ring S together with a decomposition $S = \oplus_{d \geq 0} S_d$ of S into a direct sum of abelian groups S_d such that for any $d, e \geq 0$, $S_d \cdot S_e \subseteq S_{d+e}$.

Definition 1.31 (Homogeneous Element of Degree d)**.** A *homogeneous element of degree d* is an element of S_d. Any element of S can be written uniquely as a finite sum of homogeneous elements.

Definition 1.32 (Homogeneous Ideals)**.** An ideal $\mathfrak{a} \subseteq S$ is a *homogeneous ideal* if $\mathfrak{a} = \oplus_{d \geq 0}(\mathfrak{a} \cap S_d)$. In other words, an ideal is homogeneous if it can be generated by a set of homogeneous elements.

Proposition 1.33 (Properties of Homogeneous Ideals)**.** *The finite product, arbitrary (direct) sum, and arbitrary intersection of homogeneous ideals are all homogeneous. Furthermore, a homogeneous ideal $\mathfrak{a}$ is prime if and only if, for all homogeneous elements f, g, $fg \in \mathfrak{a}$ implies $f \in \mathfrak{a}$ or $g \in \mathfrak{a}$.*

Example 1.34 (Polynomial Rings are Graded)**.** The polynomial ring $S = k[x_0, \ldots, x_n]$ is a graded ring with S_d being the *homogeneous polynomials of degree d* (that is, polynomials whose monomial terms all have degree d). For example, $x_0^2 + x_1 x_2$ is a homogeneous polynomial of degree 2 in $k[x_0, x_1, x_2]$.

Definition 1.35 (Zero Sets in Projective Space)**.** Suppose that T is a set of homogeneous polynomials in S. Then the set $\{P \in \mathbb{P}_k^n \mid f(P) = 0 \text{ for all } f \in T\}$ is well-defined and called the *zero set* of T. To see why, notice that if f is a homogeneous polynomial of degree d and $f(a_0, \ldots, a_n) = 0$, then $f(\lambda a_0, \ldots, \lambda a_n) = \lambda^d f(a_0, \ldots, a_n) = \lambda^d 0 = 0$ for any $\lambda \in k^\times$.

Definition 1.36 (Zero Set of Homogeneous Ideal)**.** Let $\mathfrak{a}$ be a homogeneous ideal of S. Then we define $Z(\mathfrak{a}) = Z(T)$, where T is the set of all homogeneous elements in $\mathfrak{a}$. Since S is a Noethering ring, any set of homogeneous elements T has a finite subset $f_1, \ldots, f_r$ such that $Z(T) = Z(f_1, \ldots, f_r)$.

Definition 1.37 (Algebraic Set in $\mathbb{P}_k^n$)**.** A subset Y of $\mathbb{P}_k^n$ is an *algebraic set* if there exists a set T of homogeneous elements of S such that $Y = Z(T)$.

Definition 1.38 (The Zariski Topology on $\mathbb{P}^n_k$)**.** Using Proposition 1.33, we see that the finite union or arbitrary intersection of algebraic sets are algebraic. Obviously $\varnothing = Z(1)$ and $\mathbb{P}^n_k = Z(0)$, so the empty set and the whole space are algebraic. Hence we may a topology, called the *Zariski topology on* $\mathbb{P}^n_k$, by defining the algebraic sets to be the closed sets.

Definition 1.39 (Projective Algebraic Varieties)**.** A *projective (algebraic) variety* is an irreducible algebraic set in $\mathbb{P}^n_k$. An open subset of a projective variety is a *quasi-projective variety.*

Definition 1.40 (Homogeneous Ideal of Projective Subset)**.** For a projective subset Y of $\mathbb{P}^n_k$, the *homogeneous ideal* of Y in S, denoted $I(Y)$, to be the ideal generated by

$$\{f \in S \mid f \text{ is homogeneous and } f(P) = 0 \text{ for all } P \in Y\}.$$

Definition 1.41 (Projective Hyperplane)**.** If $f \in S$ is a linear homogeneous polynomial (a homogeneous polynomial of degree 1), then $Z(f)$ is called a *(projective) hyperplane.* In particular, $Z(x_i)$ is denoted by H_i.

Theorem 1.42 (Covering Projective Space with Affine Spaces)**.** *For each i, let $U_i = \mathbb{P}^n_k \setminus H_i$. Then $\{U_i\}$ is an open cover of $\mathbb{P}^n_k$. Furthermore, U_i is homeomorphic to affine n-space under the homeomorphism*

$$\varphi_i : U_i \to \mathbb{A}^n \text{ given by } (a_0, \dots, a_n) \mapsto \left(\frac{a_0}{a_i}, \dots, \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i}, \frac{a_n}{a_i}\right).$$

Proof. Clearly, φ_i is bijective, so it suffices to show that it is closed and continuous. For this, $S = k[x_0, \dots, x_n]$ and $A = k[y_1, \dots, y_n]$. Let S^h be the set of homogeneous elements of S. Now, define the map $\alpha : S^h \to A$ by $\alpha(f) = f(1, y_1, \dots, y_n)$. Similarly, define $\beta : A \to S^h$ as follows: if $g \in A$ has degree d, then $\beta(g) = x_0^d g(x_1/x_0, \dots, x_n/x_0)$, which is a homogeneous polynomial of degree d.

Now let $Y \subseteq U_i$ be closed, with closure $\overline{Y}$ in $\mathbb{P}^n_k$. This is an algebraic set, so $\overline{Y} = Z(T)$ for some subset $T \subseteq S^h$. Let $T' = \alpha(T)$. Then it is easy to check that $\varphi(Y) = Z(T')$, so φ is closed. Similarly, if $W \subseteq \mathbb{A}^n_k$ is closed, then $W = Z(T')$ for some subset T' of A, and it is easy to check that $\varphi^{-1}(W) = Z(\beta(T')) \cap U_i$. Hence φ is also continuous, as desired. Therefore we are done. ☐

Since $H_i \cong \mathbb{P}^{n-1}_k$, this result formalizes the earlier idea of considering the plane $a_i = 1$ to decompose $\mathbb{P}^n_k$ as the disjoint union $\mathbb{A}^n_k \sqcup \mathbb{P}^{n-1}_k$. However, the open cover $U_1 \cup \cdots \cup U_n$ of $\mathbb{P}^n_k$ often proves to be ultimately more useful, because it breaks down projective n-space as a union composed entirely of affine n-spaces.

Corollary 1.42.1 (Decomposition of Projective Varieties)**.** *If Y is a projective (resp. quasi-projective) variety, then Y is covered by the open sets $Y \cap U_i$ for $i = 0, \dots, n$ which are each homeomorphic to affine (resp. quasi-affine) varieties by the restriction $\varphi_i|_{Y \cap U_i}$ of the mapping φ_i defined above.*

Now, we will recount many projective versions of affine results. The proof of these results usually amounts to reducing to the affine case using either the direct definition or the affine covering discussed in Theorem 1.42. Because this tactic is instructive (and because, unfairly, all of these are Hartshorne exercises instead of theorems with included proofs), I still offer proofs for them.

Theorem 1.43 (The Homogeneous Nullstellensatz)**.** *Let k be an algebraically closed field and $S = k[x_0, \dots, x_n]$ with the usual graded ring structure. Suppose that $\mathfrak{a} \subseteq S$ is a homogeneous ideal such that $Z(\mathfrak{a}) \neq \varnothing$. Then $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.*

Proof. Let $\mathfrak{a} \subseteq S$ be a homogeneous ideal with $Z(\mathfrak{a}) \neq \varnothing$. Firstly, notice that obviously $\sqrt{\mathfrak{a}} \subseteq I(Z(\mathfrak{a}))$. On the other hand, suppose that $f \in I(Z(\mathfrak{a}))$. Since $Z(\mathfrak{a}) \neq \varnothing$, either $f = 0$ or $\deg(f) > 0$. In the former case, clearly $f = 0 \in \sqrt{\mathfrak{a}}$. In the latter case, $(a_0 : a_1 : \cdots : a_n) \in \mathbb{P}^n_k$ is a zero of f if and only if $(a_0, \dots, a_n) \in \mathbb{A}^{n+1}_k$ is a zero of f (when f is considered as a map $\mathbb{A}^{n+1} \to k$). Then, by the ordinary Nullstellensatz, $f \in \sqrt{\mathfrak{a}}$ anyways. Hence in any case, $f \in I(Z(\mathfrak{a}))$ implies $f \in \mathfrak{a}$. Therefore $I(Z(\mathfrak{a})) \subseteq \mathfrak{a}$, as desired. ☐

Proposition 1.44 (Criterion for Emptiness)**.** *Suppose that $\mathfrak{a} \subseteq S$ is a homogeneous ideal. Then the following conditions are equivalent: (i) $Z(\mathfrak{a}) = \varnothing$, (ii) $\sqrt{\mathfrak{a}} = S$ or the "irrelevant maximal ideal" $S_+ = \oplus_{d>0} S_d$, and (iii) $S_d \subseteq \mathfrak{a}$ for some $d > 0$.*

Proof.
(i) ⇒ (ii): If $Z(\mathfrak{a})$ is empty, then in $\mathbb{A}_k^{n+1}$ either $Z(\mathfrak{a})$ is empty or $Z(\mathfrak{a}) = \{(0, \dots, 0)\}$. In the former case, $\sqrt{a} = I(Z(\mathfrak{a})) = k[x_0, \dots, x_n] = S$. In the latter case, $\sqrt{\mathfrak{a}} = I(Z(\mathfrak{a})) = (x_0, \dots, x_n) = S_+$.

(ii) ⇒ (iii): In either case, $\sqrt{a}$ contains S_+. Then, there exists some integer m_i such that $x_i^{m_i} \in \mathfrak{a}$ for each $i = 0, \dots, n$. Take $m = \max_i\{m_i\}$, so that $x_i^m \in \mathfrak{a}$ for each i. But then every monomial of degree $m(n+1)$ is divisible by x_i^m for some i by the Pigeonhole Principle, so $S_{m(n+1)} \subseteq \mathfrak{a}$, as desired.

(iii) ⇒ (i): Let $\mathfrak{a} \supseteq S_d$ for some $d > 0$. Then $x_i^d \in \mathfrak{a}$ for $i = 0, \dots, n$, and they have no common zeroes in $\mathbb{P}_k^n$, so $Z(\mathfrak{a}) = \varnothing$. □

Proposition 1.45 (Properties of Algebraic Sets in $\mathbb{P}_k^n$)**.**

(1) If $T_1 \subseteq T_2$ are subsets of homogeneous elements of S, then $Z(T_1) \supseteq Z(T_2)$.

(2) Conversely, if $Y_1 \subseteq Y_2$ are subsets of $\mathbb{P}_k^n$, then $I(Y_1) \subseteq I(Y_2)$.

(3) If $Y_1, Y_2 \subseteq \mathbb{P}_k^n$, then $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.

(4) For any subset $Y \subseteq \mathbb{P}_k^n$, $Z(I(Y)) = \overline{Y}$, the closure of Y.

Proof. All four are trivial. □

Theorem 1.46 (Algebro-Geometric Correspondence for Projective Space)**.** *Suppose that k is an algebraically closed field. Then there is an inclusion-reversing correspondence between types of closed subsets of $\mathbb{P}_k^n$ and types of ideals in the graded rings in $S = k[x_0, \dots, x_n]$, as follows:*

1. *Homogeneous radical ideals (other than S_+) correspond to closed subsets of $\mathbb{P}_k^n$.*
2. *Homogeneous prime ideals (other than S_+) correspond to irreducible subsets of $\mathbb{P}_k^n$.*
3. *Maximal ideals of A (other than S_+) correspond to single points of $\mathbb{P}_k^n$.*

Recall that $S_+ = \oplus_{d>0} S_d$ is the "irrelevant maximal ideal" covered in Proposition 1.44.

Proof. This follows immediately from the identical results in the affine case, as well as Theorem 1.43, Proposition 1.44, and Proposition 1.45. □

The *affine cone* is a useful tool for reducing projective subsets to affine subsets.

Definition 1.47 (The Affine Cone)**.** Let $\theta : \mathbb{A}_k^{n+1} \setminus \{(0, \dots, 0)\} \to \mathbb{P}_k^n$ be the projection map $(x_0, \dots, x_n) \mapsto (x_0 : \dots : x_n)$. Then if $Y \subseteq \mathbb{P}_k^n$, the *affine cone over* Y is the set $C(Y) = \theta^{-1}(Y) \cup \{(0, \dots, 0)\}$.

Proposition 1.48 (The Topology of Projective Space)**.**

(i) $\mathbb{P}^n$ is a Noetherian topological space.

(ii) Every algebraic set in $\mathbb{P}^n$ can be written uniquely as a finite union of irreducible algebraic sets, no one containing another. These are called its irreducible components.

(iii) Suppose that Y is a projective variety with homogeneous coordinate ring $S(Y) = S/I(Y)$. Then $\dim S(Y) = \dim Y + 1$. *In particular,* $\dim \mathbb{P}^n = n$.

(iv) A projective variety $Y \subseteq \mathbb{P}_k^n$ has dimension $n - 1$ if and only if it is the zero set of a single irreducible homogeneous polynomial f of positive degree. Y is called a hypersurface *in $\mathbb{P}_k^n$.*

Proof.
(i): By Proposition 1.45, a descending chain of irreducible closed subsets of $\mathbb{P}_k^n$ corresopnding to an ascending chain of prime ideals in $k[x_0, \dots, x_n]$, which must stabilize since $k[x_0, \dots, x_n]$ is Noetherian. Hence the descending chain of irreducible closed subsets also stabilizes, as desired.

(ii): This is an immediate corollary of Proposition 1.20 and (i).

(iii): First, recall from Corollary 1.42.1 that Y has an open cover $\{Y \cap U_i\}$ for $i = 0, \ldots, n$, and that furthermore $Y \cap U_i$ is affine for each i. Yet recall from Proposition 6.24 that $\dim Y = \sup \dim Y \cap U_i$. Since the supremum is taken over a finite set, there exists an integer i such that $\dim Y = \dim Y \cap U_i$.

Assume, without loss of generality, that $i = 0$, and let $Y' = Y \cap U_0$. Then consider the ring S_{x_0} taken by localizing S to make x_0 invertible. Then, suppose that $\frac{f}{x_0^n} \in S_{x_0}$ has degree 0. Then $\frac{f}{x_0}^n = f(1, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0})$ is the element $\alpha(f) \in A(Y')$, where α is defined for the proof of Theorem 1.42 (asuming that we take $\frac{x_i}{x_0}$ to be the ith coordinate of affine space for each i).

On the other hand, given an element $g(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}) \in A(Y_0)$, by multiplying through by x_0^d (where d is the degree of g as a polynomial in $k[\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}]$), we get a homogeneous polynomial $\beta(g)$. Yet $\beta(g)$ is naturally associated to the degree zero element $\frac{\beta(g)}{x_0^d} \in S_{x_0}$. Since these two processes are mutual inverses, they give an isomorphism of $A(Y')$ with the subring of S_{x_0} of elements of degree 0. Then $S_{x_0} \simeq A(Y')[x_0, \frac{1}{x_0}]$. Yet the transcendence degree of $\text{Frac}(A(Y')[x_0, \frac{1}{x_0}])$ is one greater than the transcendence degree of $\text{Frac}(A(Y'))$, whence by Theorem 1.24, $\dim S_{x_0} = \dim A(Y')[x_0, \frac{1}{x_0}] = \dim A(Y') + 1 = \dim Y' + 1 = \dim Y + 1$.

(iv): Let $Y \subseteq \mathbb{P}^n$ have dimension $n - 1$. Then $\dim k[Y] = \dim Y + 1 = n$. This corresponds to an n-dimensional variety Y' in $\mathbb{A}^{n+1}$. By Proposition 1.28, $I(Y')$ is principal, generated by an irreducible polynomial f. Then, it is easy to check that $Y = Z(\beta(f))$ (and clearly $\beta(f)$ must be irreducible and non-constant since Y is proper and irreducible).

Conversely, let $f \in k[x_0, \ldots, x_n]$ be a non-constant irreducible homogeneous polynomial defining an irreducible variety $Z(f)$. Its ideal (f) has height 1 by Krull's Haupidealsatz (Theorem 1.26), whence $C(Z(f))$ (which is equal to the affine zero set of f in $\mathbb{A}_k^{n+1}$) has dimension n. But then $S(Z(f)) = A(C(Z(f)))$ has dimension n, so by (iii), the projective zero set $Z(f)$ has dimension $n + 1$. □

1.5 Morphisms and Regular Maps

For the rest of this section, let k be an algebraically closed field.

Definition 1.49 (Regular at a Point)**.** Let Y be a quasi-affine variety in $\mathbb{A}_k^n$. A function $f : Y \to k$ is *regular at a point* $P \in Y$ if there is an open neighborhood U of P and polynomials $g, h \in A = k[x_1, \ldots, x_n]$ such that h is nowhere zero on U and $f = g/h$ on U. We say that f is *regular on* Y if it is regular at every point of Y.

Lemma 1.50. *A regular function $Y \to k$ is continuous when k is topologized by identifying it with $\mathbb{A}_k^1$.*

Proof. Let $f : Y \to k$ be regular. Since any proper closed set of $\mathbb{A}_k^1$ is finite, it suffices to show that the preimage of any point is closed. Yet this is easy by passing to an open cover $\{U_i\}$ of Y such that f is a ratio of polynomials on each U_i and using the fact that closedness is a local condition (see Lemma 6.12). □

Definition 1.51 (Regular at a Point, Projective)**.** Let Y be a quasi-projective variety in $\mathbb{P}_k^n$. A function $f : Y \to k$ is *regular at a point* $P \in Y$ if there is an open neighborhood U of P and polynomials $g, h \in S = k[x_0, \ldots, x_n]$, homogeneous with the same degree, such that h is nowhere zero on U and $f = g/h$ on U. We say that f is *regular on* Y if it is regular at every point of Y.

Note that the requirement that g and h are homogeneous with the same degree ensures that g/h can be viewed as a well-defined function. Again, a regular function on a quasi-projective variety is continuous.

Lemma 1.52. *Suppose that f and g are regular functions on a variety X and $f = g$ on some nonempty open subset $U \subseteq X$. Then $f = g$ everywhere.*

Proof. Let V be the set of points $P \in X$ where $f(P) = g(P)$. Now, $U \subseteq V$, and U is a nonempty open of an irreducible space, so it is dense (see Proposition 6.13). Hence, to show that $V = X$, it suffices to show that

V is closed. For this, one passes to an open cover $\{U_i\}$ of X such that f and g are ratios of polynomials on each U_i and uses the fact that closedness is a local condition (see Lemma 6.12). □

Definition 1.53 (Category of Varieties)**.** Let k be a fixed algebraically closed field. A *variety of k* is any affine, quasi-affine, projective, or quasi-projective variety. These form the objects of a category of varieties over k, whose *morphisms* are continuous maps $\varphi : X \to Y$ such that for every open set $V \subseteq Y$ and every regular function $f : V \to k$, the function $f \circ \varphi : \varphi^{-1}(V) \to k$ is regular. An *isomorphism* is, as usual, a morphism with a two-sided inverse morphism, and two varieties X and Y are called *isomorphic* if there is an isomorphism between them.

Definition 1.54 (Invariants of Varieties)**.** Let Y be a variety. Then,

(1) $\mathcal{O}(Y)$ is the *ring of all regular functions* on Y.

(2) $\mathcal{O}_{P,Y}$ (or simply $\mathcal{O}_P$), called the *local ring of P on Y*, is the ring of germs of regular functions on Y. That is, an element of $\mathcal{O}_P$ is a pair $\langle U, f\rangle$ where U is an open subset of Y containing P, and f is a regular function on U, and where $\langle U, f\rangle = \langle V, g\rangle$ if $f = g$ on $U \cap V$. This is indeed a local ring (with residue field k), as its maximal ideal $\mathfrak{m}$ is the set of germs of regular functions which vanish at P.

(3) If Y is a variety, we define the *function field* $K(Y)$ of Y as follows: an element of $K(Y)$ is an equivalence class of pairs $\langle U, f\rangle$ where U is a nonempty open subset o Y, f is a regular function U, and where we identify $\langle U, f\rangle$ and $\langle V, g\rangle$ if $f = g$ on $U \cap V$. The elements of $K(Y)$ are called *rational functions* on Y.

Notice that there are natural maps $\mathcal{O}(Y) \to \mathcal{O}_P \to K(Y)$, which are injective by Lemma 1.52.

Theorem 1.55. *Let $Y \subseteq \mathbb{A}^n_k$ be an affine variety with affine coordinate ring $A(Y)$. Then:*

(a) $\mathcal{O}(Y) \simeq A(Y)$*;*

(b) For each point $P \in Y$, let $\mathfrak{m}_P \subseteq A(Y)$ be the ideal of functions vanishing at P. Then $P \mapsto \mathfrak{m}_P$ given a 1-1 correspondence between the points of Y and the maximal ideals of $A(Y)$;

(c) For each P, $\mathcal{O}_P \simeq A(Y)_{\mathfrak{m}_P}$, and $\dim \mathcal{O}_P = \dim Y$*.*

(d) $K(Y)$ is isomorphic to the quotient field of $A(Y)$ and hence $K(Y)$ is a finitely generated extension file of k of transcendence degree $\dim Y$*.*

Proof. Page 17 of Hartshorne. □

A similar result holds for projective varieties, but we need to introduce some new notation.

Definition 1.56 (Grading Localizations of Graded Rings)**.** Suppose that S is a graded ring and T is a multiplicative subset of homogeneous elements. Then $T^{-1}(S)$ has a natural grading given by $\deg(f/g) = \deg(f) - \deg(g)$. In particular, in the case $T = S \setminus \mathfrak{p}$, we have a local graded ring $S_{\mathfrak{p}}$. The subring of elements of degree 0 in this ring is denoted $S_{(\mathfrak{p})}$, and is itself a local ring with maximal ideal $\mathfrak{p}_{\mathfrak{p}} \cap S_{(\mathfrak{p})}$. Similarly, if $f \in S$ is a homogeneous element, we denote by $S_{(f)}$ the subring of elements of degree 0 in S_f.

Theorem 1.57. *Let Y be a projective variety with homogeneous coordinate ring $S(Y)$. Then:*

(a) $\mathcal{O}(Y) = k$*;*

(b) For any point $P \in Y$, let $\mathfrak{m}_P \subseteq S(Y)$ be the ideal generated by the set of homogeneous $f \in S(Y)$ vanishing at P. Then $\mathcal{O}_P = S(Y)_{(\mathfrak{m}_P)}$.

(c) $K(Y) \simeq S(Y)_{((0))}$.

Proof. Pages 18-19 of Hartshorne. □

Lemma 1.58. *Let X be any variety, and let $Y \subseteq \mathbb{A}^n_k$ be an affine variety. A map of sets $\psi : X \to Y$ is a morphism if and only if $x_i \circ \psi$ is a regular function on X for each i, where $x_1, \dots, x_n$ are the coordinate functions on $\mathbb{A}^n_k$.*

Proof. Clearly, if ψ is a morphism, then $x_i \circ \psi$ is regular by definition of a morphism. On the other hand, if $x_i \circ \psi$ is regular, then $f \circ \psi$ is regular for any polynomial $f(x_1, \ldots, x_n)$ (since the sum and product of regular functions is regular). Now, take $V \subseteq Y$ closed; that is, $V = Z(f_1, \ldots, f_n)$ for some polynomials $f_1, \ldots, f_n \in A$. Yet, just as $V = \bigcup_{i=1}^n f_i^{-1}(0)$ implies $\psi^{-1}(V) = \bigcap_{i=1}^n \psi^{-1}(f_i^{-1}(0)) = \bigcap_{i=1}^n (f_i \circ \psi)^{-1}(0)$.

Since $\{0\}$ is closed in $\mathbb{A}_k^1$, and $\psi \circ f_i$ is regular and hence continuous, $(f_i \circ \psi)^{-1}(0)$ is closed. Hence $\psi^{-1}(V)$, as the intersection of closed sets, is closed. Therefore ψ is continuous. Furthermore, since regular functions on open subsets of Y are locally quotients of polynomials, $g \circ \psi$ is regular for any regular function g on any open subset of Y. Hence ψ is a morphism. □

Proposition 1.59. *Let X be any variety and let Y be an affine variety. Then there is a natural bijective mapping of sets $\alpha : \operatorname{Hom}(X, Y) \xrightarrow{\sim} \operatorname{Hom}(A(Y)), \mathcal{O}(X))$.*

Proof. Given a morphism $\varphi : X \to Y$, φ carries regular functions on Y to regular functions on X. Hence φ induces a map $\mathcal{O}(Y)$ to $\mathcal{O}(X)$, and since $\mathcal{O}(Y) \simeq A(Y)$, we have an induced map $\alpha(\varphi) : A(Y) \to \mathcal{O}(X)$.

Conversely, suppose we are given a morphism $\psi : A(Y) \to \mathcal{O}(X)$ of k-algebras. Since Y is an affine variety, $Y \subseteq \mathbb{A}_k^n$, so that $A(Y) = k[x_1, \ldots, x_n]/I(Y)$. Let $\overline{x}_i$ be the image of x_i in $A(Y)$, and consider the elements $\xi_i = \psi(\overline{x}_i) \in \mathcal{O}(X)$. These are global functions on X, so we can use them to define a mapping $\beta(\psi) : X \to \mathbb{A}_k^n$ by $\beta(\psi)(P) = (\xi_1(P), \ldots, \xi_n(P))$. Furthermore, the image of $\beta(\psi)$ is contained in P, since we may easily check that for any $f \in I(Y)$, $f(\psi(P)) = 0$ (and of course $Y = Z(I(Y))$ as Y is closed). Therefore, $\beta(\psi)$ is an induced map $X \to Y$.

Now that we have maps $\alpha : \operatorname{Hom}(X, Y) \to \operatorname{Hom}(A(Y)), \mathcal{O}(X))$ and $\beta : \operatorname{Hom}(A(Y)), \mathcal{O}(X)) \to \operatorname{Hom}(X, Y)$, it suffices to show these are mutual inverses. Yet this is simple and left as an exercise to the reader, as is showing naturality. □

Corollary 1.59.1. *If X, Y are two affine varieties, then X and Y are isomorphic if and only if $A(X)$ and $A(Y)$ are isomorphic k-algebras.*

Now that we have a robust notion of isomorphism, we may define what it means for an arbitrary variety to be "affine", and explore why affine varieties form a basis for all other varieties.

Definition 1.60 ((Quasi)-Affine)**.** An arbitrary variety is called "affine" if it is isomorphic to an affine variety. Similarly, an arbitrary variety is called "quasi-affine" if it is isomorphic to a quasi-affine variety.

Lemma 1.61. *Let Y be a hypersurface in $\mathbb{A}_k^n$ which is the zero set of $f \in k[x_1, \ldots, x_n]$. Then $\mathbb{A}_k^n \setminus Y$ is isomorphic to the hypersurface H in $\mathbb{A}^{n+1}$ which is the zero set of $x_{n+1} f = 1$. In particular, $\mathbb{A}_k^n \setminus Y$ is affine with affine coordinate ring $k[x_1, \ldots, x_n]_f$.*

Proposition 1.62. *On any variety Y, there is a base for the topology consisting of open affine subsets.*

Proof. Choose a point $P \in Y$ and an open neighborhood U of Y. It suffices to show that there is an open affine neighborhood V of P contained in U. Now, as an open subset of a variety, U is a variety, so we may assume that $U = Y$. Furthermore, since by Corollary 1.42.1 any variety is covered by open quasi-affine varieties, we may assume that Y is quasi-affine in $\mathbb{A}_k^n$.

Now, let $Z = \overline{Y} \setminus Y$. I claim that this is a closed subset of $\mathbb{A}_k^n$. To see why, notice that Y is contained in an affine variety V, which is closed in $\mathbb{A}_k^n$, so $\overline{Y} \subseteq V$. Then $\overline{Y} \setminus Y = \overline{Y} \cap (V \setminus Y)$ is closed in V as it is the intersection of two closed subsets of V. But then $\overline{Y} \setminus Y$ is the closed subset of a closed subspace V of $\mathbb{A}_k^n$ and hence closed in $\mathbb{A}_k^n$.

Hence Z has a corresponding ideal $I(Z)$. Since $P \notin Z$, there exists $f \in I(Z)$ such that $f(P) \neq 0$. Let H be the hypersurface $f = 0$ in $\mathbb{A}_k^n$. Then since $P \notin H$, $P \in Y \setminus (Y \cap H)$, which is an open subset of Y since H is closed in $\mathbb{A}_k^n$ whence $Y \cap H$ is closed in Y. Furthermore, $Y \setminus Y \cap H$ is a closed subset of $\mathbb{A}^n \setminus H$, which is affine by Lemma 1.61. Hence $Y \setminus Y \cap H$ is a closed subset of an affine variety and hence is affine. Then, by Corollary 1.20.1, we may choose an open affine variety contained in $Y \setminus Y \cap H$, which is the desired affine neighborhood of P. Therefore we are done. □

1.6 Rational Maps

Definition 1.63 (Separatedness)**.** A variety X is called *separated* if for any other variety Y and any morphisms $\varphi, \psi : Y \rightrightarrows X$, the set of points where φ and ψ agree is a closed subset of Y.

In turns out that all varieties are separated. Later, when we enlarge our definition of "variety" to be a specific type of scheme, we will ensure that separatedness is part of the definition of a variety.

Proposition 1.64 (The Diagonal Condition)**.** *Consider the maps $\pi_1, \pi_2 : X \times X \rightrightarrows X$ given by $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$ respectively. Let $\Delta(X)$ be the set of points where π_1 and π_2 agree. Then X is separated if and only if $\Delta(X)$ is closed in $X \times X$.*

Proof. The "only if" direction follows immediately from the definition of separatedness. For "if", suppose that $\Delta(X)$ is closed in $X \times X$. Then take $\varphi, \psi : Y \rightrightarrows X$; this induces a map $(\varphi, \psi) : Y \to X \times X$. Since φ, ψ are continuous, so is (φ, ψ). The set of points in Y where φ and ψ agree is precisely the preimage of $\Delta(X)$, which is closed if $\Delta(X)$ is closed since (φ, ψ) is continuous. Hence we are done. □

Lemma 1.65. *Let X be an affine variety. Then X is separated.*

Proof. Take any variety Y and morphisms $\varphi, \psi : Y \rightrightarrows X$. By assumption, $X \subseteq \mathbb{A}_k^n$ for some algebraically closed field k and some positive integer n. Now, the set of points where φ and ψ agree is equal to the set of points where $\iota \circ \varphi$ and $\iota \circ \psi$ agree. Therefore, assume that φ, ψ map into $\mathbb{A}_k^n$. Notice that $\varphi(y) = \psi(y)$ if and only if $x_i(\varphi(y)) = x_i(\psi(y))$ (where x_i is the ith coordinate function) for each i. Hence $\{y \in Y \mid \varphi(y) = \psi(y)\}$ is the zero locus of $\{x_1(\varphi(y)) - x_1(\psi(y)), \ldots, x_n(\varphi(y)) - x_n(\psi(y))\}$ and hence closed. □

Here is another lemma extending the behavior along the usual theme "closedness is a local property".

Lemma 1.66. *Let X be a variety such that for all $x, y \in X$, there is an open affine U containing both x and y. Then X is separated.*

Proof. Consider two functions $\varphi, \psi : Y \rightrightarrows X$ and let $Z = \{y \in Y \mid \varphi(y) = \psi(y)\}$. Take $z \in \overline{Z}$; in suffices to show $z \in Z$. In other words, it suffices to show $\varphi(z) = \psi(z)$.

By assumption, there is an open affine $V \subseteq X$ containing $\varphi(z)$ and $\psi(z)$. Let $U = \varphi^{-1}(V) \cap \psi^{-1}(V)$; this is an open neighborhood of z. Then $\varphi|_U, \psi|_U$ map into affine varieties, whence $Z \cap U$ is closed. Since $Z \cap U$ is closed, $Z \cap U = \overline{Z \cap U} = \overline{Z} \cap U$, so $z \in \overline{Z} \cap U$ implies $z \in Z \cap U$ implies $z \in Z$, as desired. □

Corollary 1.66.1. *Using our current definition of "variety" (that is, any variety is affine, quasi-affine, projective, or quasi-projective), varieties are separated.*

Lemma 1.67 (Morphisms Equal on Nonempty Opens are Equal)**.** *Let X and Y be varieties and $\varphi, \psi : X \rightrightarrows Y$ be morphisms. Suppose there is a nonempty open subset $U \subseteq X$ such that $\varphi|_U = \psi|_U$. Then $\varphi = \psi$.*

Proof. The set upon which $\varphi = \psi$ is closed (by separatedness) and dense (by hypothesis), so equal to X. □

Definition 1.68 (Rational Map)**.** Let X, Y be varieties. A *rational map* $\varphi : X \to Y$ is an equivalence class of pairs $\langle U, \varphi_U \rangle$ where U is a nonempty open subset of X and φ_U is a morphism of U to$Y$, and where $\langle U, \varphi_U \rangle$ and $\langle V, \varphi_V \rangle$ are equivalent if φ_U and φ_V agree on $U \cap V$. The rational map φ is *dominant* if for some (and hence every) pair $\langle U, \varphi_U \rangle$, the image of φ_U is dense in Y.

Notice that we require Lemma 1.67 to see that this is indeed an equivalence relation.

Definition 1.69 (Birational Map)**.** A *birational map* $\varphi : X \to Y$ is a rational map with an inverse rational map; that is, a rational map $\psi : Y \to X$ such that $\psi \circ \varphi = \mathrm{id}_X$ and $\varphi \circ \psi = \mathrm{id}_Y$ (as rational maps).

Now, we will explore how birational maps form a very natural notion of "morphism". For this, we demonstrate that the category of varieties over k, with morphisms being dominant rational maps, is equivalent to the (opposite) category of finitely generated field extensions of k.

Proposition 1.70. *Any dominant rational map $\varphi : X \to Y$ induces a homomorphism of K-algebras from $K(Y)$ to $K(X)$.*

Proof. Suppose that φ is represented by, say $\langle U, \varphi_U \rangle$. Suppose we have a rational function $f \in K(Y)$ represented by $\langle V, f \rangle$. Then since $\varphi_U(U)$ is dense in Y, $\varphi_U^{-1}(V)$ is a nonempty open subset of X, so $f \circ \varphi_U$ is a regular function on $\varphi_U^{-1}(V)$. This allows us to define a map $\psi : K(Y)$ to $K(X)$ by $f \mapsto f \circ \varphi_U$, which is clearly a homomorphism. □

Theorem 1.71. *For any two varieties X and Y, the above construction gives a bijection between (i) the set of dominant rational maps from X to Y, and (ii) the set of k-algebra homomorphisms from $K(Y)$ to $K(X)$.*

Proof. It suffices to construct an inverse to the above construction. For this, take a k-algebra homomorphism $\theta : K(Y) \to K(X)$. By Proposition 1.62, Y is covered by open affine varieties, so we may assume that Y is an affine variety. Let $A(Y)$ be an affine coordinate ring, and let $y_1, \ldots, y_n$ be generators for $A(Y)$ as a k-algebra. Then $\theta(y_1), \ldots, \theta(y_n)$ are rational functions on X. We can find an open set $U \subseteq X$ such that the functions $\theta(y_i)$ are all regular on U. Then θ defines an injective homomorphism of k-algebras $A(Y) \to \mathcal{O}(U)$. By Proposition 1.59, this corresponds to a morphism $\varphi : U \to Y$, which gives a dominant rational map $X \to Y$. It is easy to see this construction is the inverse of the one discussed in Proposition 1.70. □

Theorem 1.72. *The above correspondence gives a contravariant equivalence of categories of the category of varieties and dominant rational maps with the category of finitely generated field extensions of k.*

Proof. It suffices to show that (1) for any variety Y, $K(Y)$ is finitely generated over k, and (2) conversely, if K/k is a finitely-generated field extension, then $K = K(Y)$ for some Y. For (1), if Y is a variety, then $K(Y) = K(U)$ for any open affine subset, so we may assume Y is affine. Then $K(Y)$ is a finitely generated field extension of k by Theorem 1.55. Conversely, let K be a finitely generated field extension of k, generated by $y_1, \ldots, y_n$. Let B be the sub-k-algebra of K generated by $y_1, \ldots, y_n$; this is a domain since it is a subalgebra of K. Then B is a quotient of the polynomial ring $A = k[x_1, \ldots, x_n]$, so $B \simeq A(Y)$ for some variety Y in $\mathbb{A}_k^n$. Then $K \simeq K(Y)$, so (2) is also true. Hence we are done. □

Corollary 1.72.1. *For any two varieties X, Y the following conditions are equivalent:*

(i) *X and Y are birationally equivalent;*

(ii) *there are open subsets $U \subseteq X$ and $V \subseteq Y$ with U isomorphic to V,*

(iii) *$K(X) \simeq K(Y)$ as k-algebras.*

Proof. The only one which does not follow from the above theorem or the definition is (i) $\Rightarrow$ (ii), but even this follows immediately by definition-shuffling. □

2 Schemes

Right now, our notion of variety has three major limitations. Firstly, most of our nontrivial results require us to work over an algebraically closed field. This poses a major issue, especially for number theorists and arithmetic geometers in particular. Secondly, our varieties currently need an embedding into affine or projective space, but it would be nice to have some kind of abstract variety. In general, it is often useful to define mathematical objects which only look locally like objects we are familiar with; for example, after working through basic real analysis a natural next step is to define a manifold, which locally looks like $\mathbb{R}^n$. Finally, only being able to work with irreducible algebraic sets can cause issues; for example, with our current definition, the intersection of two varieties is not necessarily a variety. Schemes, invented by Grothendieck, generalize varieties (in a precise way which we will discuss later) in a way that solves all three issues.

2.1 Sheaves and Morphisms

Definition 2.1 (Topological Category)**.** Suppose that X is a topological space. Then $\mathbf{Top}(X)$ is the category whose objects are the open subsets of X and whose morphisms are inclusion maps $U \hookrightarrow V$.

Definition 2.2 (Presheaf)**.** Let X be a topological space. A *presheaf* $\mathscr{F}$ of abelian groups (resp. sets, rings) on X is a contravariant functor from the category $\mathbf{Top}(X)$ to the category of abelian groups (resp. sets, rings). That is, a presheaf $\mathscr{F}$ of abelian groups (resp. sets, rings) consists of the data:

(1) an abelian group (resp. set, ring) $\mathscr{F}(U)$ for every open subset $U \subseteq X$, and

(2) a abelian group homomorphism (resp. function, ring homomorphism) $\rho_{VU} : \mathscr{F}(V) \to \mathscr{F}(U)$, called a "restriction map", for every inclusion $U \subseteq V$ of open subsets of X,

such that (i) $\rho_{UU} = \mathrm{id}_{\mathscr{F}(U)}$ for each open $U \subseteq X$ and (ii) if $U \subseteq V \subseteq W$ is a chain of open subsets, then $\rho_{WU} = \rho_{VU} \circ \rho_{WV}$. Notice that, in particular, this implies that $\mathscr{F}(\varnothing) = 0$.

Definition 2.3 (Section and Restriction)**.** An element of $\mathscr{F}(U)$ is a *section* of the presheaf $\mathscr{F}$ over the open set U. Often, if $s \in \mathscr{F}(U)$ and $V \subseteq U$, we denote $\rho_{UV}(s)$ by $s|_V$, as if we are "restricting" s to V.

Definition 2.4 (Sheaf)**.** A presehaf $\mathscr{F}$ on a topological space X is a *sheaf* if it satisfies the following axioms:

(1) If U is an open set, if $\{V_i\}$ is an open covering of U, and $s \in \mathscr{F}(U)$ is an element such that $s|_{V_i} = 0$ for all i, then $s = 0$ (uniqueness axiom).

(2) If U is an open set, if $\{V_i\}$ is an open covering of U, and if we have elements $s_i \in \mathscr{F}(V_i)$ for each i, with the property that for each i, j, $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, then there is an element $s \in \mathscr{F}(U)$ such that $s|_{V_i} = s_i$ for each i (gluing axiom).

Notice that by the uniqueness axiom, the section obtained by the gluing axiom is necessarily unique.

Example 2.5 (Sheaf of Regular Functions)**.** Let X be a variety over the field k. For each open set $U \subseteq X$, let $\mathcal{O}(U)$ be the ring of regular functions $U \to k$, and for each $V \subseteq U$, let $\rho_{UV} : \mathcal{O}(U) \to \mathcal{O}(V)$ be given by restriction of functions. Then $\mathcal{O}$ is a sheaf of rings on X, called the *sheaf of regular functions on* X.

Example 2.6 (Constant Sheaf)**.** Let X be a topological space and A an abelian group. Then the *constant sheaf* $\mathcal{O}$ *on* X is given by assigning $\mathcal{O}(U) = A$ for each open subset $U \subseteq X$ and letting $\rho_{UV} : A \to A$ be the identity map for each $V \subseteq U$.

Definition 2.7 (Stalk)**.** If $\mathscr{F}$ is a presheaf on X, and $x \in X$ is a point, then the *stalk* $\mathscr{F}_x$ of $\mathscr{F}$ at x is the direct limit $\varinjlim \mathscr{F}(U)$ of the groups $\mathscr{F}(U)$ for all open sets U containing x via the restriction maps ρ. Elements of the stalk are called *germs of sections of* $\mathscr{F}$ *at* x. Explicitly, an element of $\mathscr{F}_x$ is an equivalence class $\langle U, s\rangle$, where U is an open neighborhood of x, $s \in \mathscr{F}(U)$, and $\langle U, s\rangle = \langle V, t\rangle$ iff there is an open neighborhood $W \subseteq U, V$ of x such that $s|_W = t|_W$. Given a section $s \in \mathscr{F}(U)$, we define the image s_x of s in $\mathscr{F}_x$ to be the equivalence class which $\langle U, s\rangle$ falls into.

Example 2.8. The stalk $\mathcal{O}_P$ of the sheaf of regular functions of a variety X is the local ring of P on X.

Stalks are remarkably useful, because the sheaf axioms imply that sections are determined by their local behavior. In particular, sections are determined by their stalks:

Lemma 2.9. *Suppose $\mathscr{F}$ is a sheaf on X, and $s, t \in \mathscr{F}(U)$ satisfy $s_x = t_x$ in $\mathscr{F}_x$ for all $x \in U$. Then $s = t$.*

Proof. Take $x \in U$. Since $s_x = t_x$, there exists some neighborhood W_x of x such that $s|_{W_x} = t|_{W_x}$. That is, $(s-t)|_{W_x} = 0$. But the collection $\{W_x\}$ covers U, so by the uniqueness axiom $s - t = 0$ whence $s = t$. □

Definition 2.10 (Morphism of (Pre)Sheaves)**.** A *morphism of (pre)sheaves* $\varphi : \mathscr{F} \to \mathscr{G}$ is simply a natural transformation $\mathscr{F} \to \mathscr{G}$. That is, φ consists of morphisms $\varphi_U : \mathscr{F} \to \mathscr{G}(U)$ for each open set U, such that for each inclusion $V \subseteq U$, the diagram

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{\varphi_U} & \mathscr{G}(U) \\ \rho_{UV}\downarrow & & \downarrow\rho'_{UV} \\ \mathscr{G}(V) & \xrightarrow[\varphi_V]{} & \mathscr{G}(V) \end{array}$$

commutes (where ρ_{UV} is the restriction map $U \to V$ in $\mathscr{F}$ and ρ'_{UV} is the restriction map $U \to V$ in $\mathscr{G}$). An *isomorphism* is a morphism of sheaves with a two-sided inverse.

Notice that any morphism $\varphi : \mathscr{F} \to \mathscr{G}$ induces a morphism $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ on the stalks for any point $x \in X$. It turns out that we can check if two maps are equal by simply looking at these stalk maps:

Lemma 2.11 (Morphisms Equal on Stalks are Equal)**.** *Suppose that $\varphi, \psi : \mathscr{F} \to \mathscr{G}$ satisfy $\varphi_x = \psi_x$ for each $x \in X$. Then $\varphi = \psi$.*

Proof. This follows immediately from Lemma 2.9. □

Similarly, we can tell if a map is an isomorphism by looking at these stalk maps:

Proposition 2.12 (Isomorphism of Sheaves is Isomorphism on Stalks)**.** *Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on a topological space X. Then φ is an isomorphism if and only if the induced map on the stalk $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ is an isomorphism for every $x \in X$.*

Proof. Because the map $\mathscr{F} \mapsto \mathscr{F}_x$ is functorial in $\mathscr{F}$ for any point $x \in X$, it is clear that if φ has a two-sided inverse so does φ_x for any $x \in X$. Therefore, assume that φ_x is an isomorphism for all $x \in X$.

To show that φ is an isomorphism, it suffices to show that $\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is an isomorphism for each open $U \subseteq X$ (since then we may define $\psi : \mathscr{G} \to \mathscr{F}$ by $\psi_U = \varphi_U^{-1}$, and this is clearly natural in U). Therefore, it suffices to show that φ_U is injective and surjective.

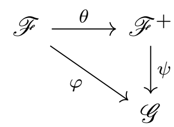
For the former, suppose $s \in \mathscr{F}(U)$ satisfies $\varphi(s) = 0$. This implies that $\varphi(s)_x = 0$ for every $x \in U$. Yet $\varphi(s)_x = \varphi_x(s_x)$, so $\varphi_x(s_x) = 0$ for all $x \in U$. Since φ_x is injective by hypothesis, $s_x = 0$ for all $x \in U$. Then by Lemma 2.9, $s = 0$. For the latter, take $t \in \mathscr{G}(U)$. Since φ_x is surjective for all x, for each $x \in U$ we can find $s_x \in \mathscr{F}_x$ such that $\varphi_x(s_x) = t_x$. Let s_x be represented by a section $s(x)$ on a neighborhood V_x of P. Because $\varphi_x(s_x) = t_x$, there must exist a neighborhood $W_x \subseteq V_x$ upon which $\varphi(s(x))|_{W_x} = t|_{W_x}$. Replace $s(x)$ with $s(x)|_{W_x}$ for all x, so that $\varphi(s(x)) = t|_{W_x}$ for each $x \in U$. Now, I claim that we can glue the $s(x)$ together into a section $s \in \mathscr{F}(U)$ using the gluing axiom. For this, notice that

(1) The collection $\{W_x\}_{x \in U}$ covers U, and we have a section $s(x) \in \mathscr{F}(W_x)$ for each $x \in U$.

(2) Suppose that x and y are distinct points. Then $s(x)|_{W_x \cap W_y}$ and $s(y)|_{W_x \cap W_y}$ are both sent by φ to $t|_{W_x \cap W_y}$, whence by injectivity they are equal. Therefore, the sections are compatible.

Hence we may apply the gluing axiom to get a section $s \in \mathscr{F}(U)$ such that $s|_{W_x} = s(x)$ for each $x \in U$. I claim that $\varphi(s) = t$. For this it suffices to show $\varphi(s) - t = 0$, and indeed this follows by the uniqueness axiom, as $(\varphi(s) - t)|_{W_x} = \varphi(s)|_{W_x} - t|_{W_x} = \varphi(s(x)) - t|_{W_x} = 0$ for each $x \in U$. Hence we are done. □

Theorem 2.13 (Sheafification)**.** *Given a presheaf $\mathscr{F}$, there is a sheaf $\mathscr{F}^+$ and a morphism $\theta : \mathscr{F} \to \mathscr{F}^+$ with the following universal property: if $\mathscr{G}$ is a sheaf and $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism, then there is a unique morphism $\psi : \mathscr{F}^+ \to \mathscr{G}$ such that the following diagram commutes:*



The pair $(\mathscr{F}^+, \theta)$ is unique up to unique isomorphism, so it may reasonably be called ***the*** sheafification *of $\mathscr{F}$. Furthermore, $\theta_x : \mathscr{F} \to \mathscr{F}^+_x$ is an isomorphism for all $x \in X$. Finally, if $\mathscr{F}$ is already a sheaf, then θ is an isomorphism, so $\mathscr{F} \simeq \mathscr{F}^+$.*

Proof. Define a presheaf $\mathscr{F}^+$ as follows:

(1) For any open set U, $\mathscr{F}^+(U)$ is the set of functions $s : U \to \coprod_{x\in U} \mathscr{F}_x$ such that (i) $s(x) \in \mathscr{F}_x$ for each x, and (ii) for each $x \in U$ there is a neighborhood $V \subseteq U$ of x and $t \in \mathscr{F}(V)$ such that for all $y \in V$, $s(y) = t_y$. Note that $\mathscr{F}^+(U)$ naturally has an abelian group or ring structure if $\mathscr{F}$ is a sheaf of abelian groups or rings.

(2) Given an inclusion of open sets $V \subseteq U$, we define the restriction map $\rho_{UV} : \mathscr{F}^+(U) \to \mathscr{F}^+(V)$ as the usual restriction of functions.

Firstly, we will verify that $\mathscr{F}^+$ is a sheaf. The uniqueness axiom is simple: if $\{V_i\}$ covers U and $s \in \mathscr{F}(U)$ satisfies $s|_{V_i} = 0$ for each i, then clearly $s = 0$; it restricts to the zero function on a cover of U, so it is the zero function. The gluing axiom is similarly simple: suppose $\{V_i\}$ covers U and $s_i \in \mathscr{F}(V_i)$ satisfy the compatibility requirement. Then clearly we can glue together the functions s_i into a function $s : U \to \coprod_{x\in U} \mathscr{F}_x$; since properties (i) and (ii) are both local, they also follow so $s \in \mathscr{F}(U)$, as desired.

Next, we define the map θ. Let θ_U be given by sending each section $s \in \mathscr{F}(U)$ to the function $u \mapsto s_u$. Clearly the output function satisfies (i) and (ii), so θ_U is indeed a morphism $\mathscr{F}(U) \to \mathscr{F}^+(U)$, and it is easy to check that it is natural, so θ is indeed a morphism $\mathscr{F} \to \mathscr{F}^+$. Next, we will show that θ_x is an isomorphism for each x. For this, we will define an inverse map κ_x. Namely, take $s_x \in \mathscr{F}^+_x$ with representative (U, s), where U is a neighborhood of x and $s \in \mathscr{F}^+(U)$. By (ii), there is a neighborhood V and $t \in \mathscr{F}(V)$ such that for all $s(y) = t_y$. This t is necessarily unique by Lemma 2.9, and we define $\kappa_x(s_x) = t$. It is easy to check that this is a well-defined map, and furthermore that it is the two-sided inverse of θ_x, as desired.

Notice that this immediately implies that if $\mathscr{F}$ is already a sheaf, then θ is an isomorphism by Proposition 2.12. It remains to show the described universal property. First, we will show uniqueness: suppose that ψ, ψ' satisfy $\varphi = \psi \circ \theta$ and $\varphi = \psi' \circ \theta$. Then, by taking stalks, $\varphi_x = \psi_x \circ \theta_x$ and $\varphi_x = \psi'_x \circ \theta_x$. But then $\psi_x = \varphi_x \circ \theta_x^{-1} = \psi'_x$ (since θ_x is an isomorphism), whence by Lemma 2.11, $\psi = \psi'$. Next, we will show existence. Suppose that we have a morphism of presheaves $\varphi : \mathscr{F} \to \mathscr{G}$ and $\mathscr{G}$ is a sheaf. Now, given $f \in \mathscr{F}^+(U)$, define $\psi'(f)$ to be the composition of f with the natural map

$$\coprod_{x\in U} \varphi_x : \coprod_{x\in U} \mathscr{F}_x \to \coprod_{x\in U} \mathscr{G}_x.$$

Now, notice that $\psi'(f)$ satisfies properties (i) and (ii) for $\mathscr{G}$, so $\psi'(f) \in \mathscr{G}^+(U)$. Hence ψ' is a morphism $\mathscr{F}^+ \to \mathscr{G}^+$. Yet the map $\theta' : \mathscr{G} \to \mathscr{G}^+$ is an isomorphism by our above work, so if we define $\psi = \theta'^{-1} \circ \psi'$ we get a map $\mathscr{F}^+ \to \mathscr{G}$, as desired. Now, we want to prove that $\varphi = \psi \circ \theta$. For this, it suffices by Lemma 2.11 to check that the stalks are equal; yet this is easy using the definition.

Finally, uniqueness of $(\mathscr{F}, \theta)$ follows from the universal property immediately via the usual argument. □

Definition 2.14 (Presheaf Kernel, Presheaf Cokernel, Presheaf Image)**.** Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves. Then, we define the following presheaves:

1. The *presheaf kernel of* φ is the presheaf given by $U \mapsto \ker(\varphi_U)$.

2. The *presheaf cokernel of* φ is the presheaf given by $U \mapsto \operatorname{coker}(\varphi_U)$.

3. The *presheaf image of* φ is the presheaf given by $U \mapsto \operatorname{im}(\varphi_U)$.

Proposition 2.15. *Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves. Then the presheaf kernel of φ is a sheaf.*

Proof. Let $\mathscr{K}$ be the presheaf kernel of φ. Take an open set $U \subseteq X$ and $s \in \mathscr{K}(U)$. Let $\{U_i\}$ be an open cover of U, and suppose $s|_{U_i} = 0$ for all i. Since $\mathscr{K}(U) \subseteq \mathscr{F}(U)$, s is also naturally an element of $\mathscr{F}(U)$, and $\mathscr{F}$ satisfies the uniqueness axiom by hypothesis, $s = 0$. Hence $\mathscr{K}$ satisfies the uniqueness axiom.

Now, it suffices to show that $\mathscr{K}$ satisfies the gluing axiom. Let $\{U_i\}$ be an open cover of U, and suppose we have elements $s_i \in \mathscr{K}(U_i)$ for each i such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for each i, j. Now, since $\mathscr{F}$ satisfies the gluing axiom, there exists $s \in \mathscr{F}(U)$ such that $s|_{U_i} = s_i$ for each i. Therefore, it suffices to show that $s \in \mathscr{K}(U)$. For this, consider $\varphi_U(s)$. Yet notice that $\{U_i\}$ is an open cover of U such that $\varphi_U(s)|_{U_i} = \varphi_{U_i}(s|_{U_i}) = \varphi_{U_i}(s_i) = 0$ for each i, so by the uniqueness axiom for $\mathscr{G}$, $\varphi_U(s) = 0$. Hence $s \in \mathscr{K}(U)$. Therefore, $\mathscr{K}$ also satisfies the gluing axiom, and we are done. □

Definition 2.16 (Kernel, Cokernel, Image)**.** Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves. Then, the *kernel of* φ, denoted $\ker \varphi$, is simply the presheaf kernel of φ. However, the *cokernel of* φ, denoted $\operatorname{coker} \varphi$, is the sheafification of the presheaf cokernel of φ, and the *image of* φ, denoted $\operatorname{im} \varphi$, is the sheafification of the presheaf image of φ. Notice that the latter two must be sheafified to ensure they are sheaves.

Definition 2.17 (Subsheaf)**.** A *subsheaf* of a sheaf $\mathscr{F}$ is a sheaf $\mathscr{F}'$ such that $\mathscr{F}'(U)$ is a subgroup (resp. subring, subset) of $\mathscr{F}(U)$ and, similarly, the restriction maps of $\mathscr{F}'$ are restrictions of the restriction maps of $\mathscr{F}$. In particular, this implies that $\mathscr{F}'_x$ is a subgroup (resp. subring, subset) of $\mathscr{F}_x$ for all $x \in X$.

Notice if $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, then $\ker \varphi$ is a subsheaf of $\mathscr{F}$ and $\operatorname{im} \varphi$ is a subsheaf of $\mathscr{G}$.

Definition 2.18 (Injectivity and Surjectivity)**.** A morphism of sheaves $\mathscr{F} \to \mathscr{G}$ is *injective* if $\ker \varphi = 0$. Thus φ is injective if and only if φ_U is injective for each open set $U \subseteq X$. On the other hand, a morphism $\mathscr{F} \to \mathscr{G}$ is *surjective* if the natural map* $\psi : \operatorname{im} \varphi \to \mathscr{G}$ is an isomorphism, but this does not necessarily imply that φ_U is surjective for each U.

*If it is not clear what this natural map is, see the proof of part (2) of Proposition 2.25, where it is explicitly constructed for the purpose of exposition.

Definition 2.19 (Exact Sequence of Sheaves)**.** A sequence $\cdots \to \mathscr{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathscr{F}^i \xrightarrow{\varphi^i} \mathscr{F}^{i+1} \to \cdots$ of sheaves and morphisms is *exact* if at each stage $\ker \varphi^i = \operatorname{im} \varphi^{i-1}$. For example, $0 \to \mathscr{F} \xrightarrow{\varphi} \mathscr{G}$ is exact if and only if φ is injective, and $\mathscr{F} \xrightarrow{\varphi} \mathscr{G} \to 0$ is exact if and only if φ is surjective.

Definition 2.20 (Quotient Sheaf)**.** Let $\mathscr{F}'$ be a subsheaf of a sheaf $\mathscr{F}$. The *quotient sheaf* $\mathscr{F}/\mathscr{F}'$ is the sheafification of the presheaf $U \to \mathscr{F}(U)/\mathscr{F}'(U)$. Notice that $(\mathscr{F}/\mathscr{F}')_x = \mathscr{F}_x/\mathscr{F}'_x$ for any $x \in X$.

Definition 2.21 (Direct Image, Inverse Image)**.** Let $f : X \to Y$ be a continuous map of topological spaces.

(1) For any sheaf $\mathscr{F}$ on X, we define the *direct image* or *pushforward* sheaf $f_*\mathscr{F}$ on Y by $(f_*\mathscr{F})(V) = \mathscr{F}(f^{-1}(V))$ for any open set $V \subseteq Y$.

(2) For any sheaf $\mathscr{G}$ on Y, we define the *inverse image* or *pullback* sheaf $f^{-1}(\mathscr{G})$ on X to be the sheafification of the presheaf $U \mapsto \lim_{V \supseteq f(U)} \mathscr{G}(V)$, where U is any open set in X, and the limit is taken over all open sets V of Y containing $f(U)$.

Of course, if f is an open map (that is, the image of an open set is open), then computing (2) is easy; however, this is not in general the case for arbitrary continuous maps.

Definition 2.22 (Restriction)**.** Suppose $Z \subseteq X$. Then, if $\iota : Z \hookrightarrow X$ is the inclusion map, and $\mathscr{F}$ is a sheaf on X, $\iota^{-1}\mathscr{F}$ is called the *restriction* of $\mathscr{F}$ to Z and denoted $\mathscr{F}|_Z$. Notice that $(\mathscr{F}|_Z)_x = \mathscr{F}_x$ for any $x \in Z$.

Definition 2.23 (Direct Sum)**.** If $\mathscr{F}$ and $\mathscr{G}$ are sheaves on X, then the presheaf $U \mapsto \mathscr{F}(U) \oplus \mathscr{G}(U)$ is a sheaf, which we call the *direct sum* of $\mathscr{F}$ and $\mathscr{G}$.

Definition 2.24 (Sheaf Hom)**.** If $\mathscr{F}$ and $\mathscr{G}$ are sheaves of abelian groups on X, then for any open set $U \subseteq X$, the set $\operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ has the natural structure of an abelian group. Hence $U \mapsto \operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ is a presheaf, and indeed a sheaf, called the *sheaf of local morphisms of* $\mathscr{F}$ *into* $\mathscr{G}$ or "sheaf hom" and denoted $\mathscr{H}om(\mathscr{F}, \mathscr{G})$.

The remainder of this section is mainly composed of solutions to exercises from Hartshorne, and lists some useful conditions and propositions. These are mainly either technical criteria which make calculations simpler, or sanity checks that properties which we are familiar with hold in the setting of sheaves as well.

First, we begin by discussing stalks, which make computations of all sorts significantly easier:

Proposition 2.25 (Stalks, Kernels and Images)**.**

1. *For any morphism of sheaves* $\varphi : \mathscr{F} \to \mathscr{G}$ *of* X*,* $(\ker \varphi)_x = \ker(\varphi_x)$ *and* $(\operatorname{im} \varphi)_x = \operatorname{im}(\varphi_x)$*.*
2. φ *is injective (resp. surjective) if and only if the induced map on the stalks* φ_x *is injective (resp. surjective) for all* $x \in X$*.*
3. *A sequence* $\cdots \to \mathscr{F}^{i-1} \stackrel{\varphi^{i-1}}{\to} \mathscr{F}^i \stackrel{\varphi^i}{\to} \mathscr{F}^{i+1} \to \cdots$ *of sheaves is exact if and only if for each* $x \in X$ *the corresponding sequence of stalks is exact as a sequence of abelian groups.*

Proof.
(1): Choose a point $x \in X$ and an element $s_x \in (\ker \varphi)_x$. Choose an pair $\langle U, s \rangle$ representing s_x. Then $\varphi_U(s) = 0$, so in particular $\varphi_U(s)_x = 0 \in \mathscr{G}_x$. Then, by the following commutative diagram,

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{\varphi_U} & \mathscr{G}(U) \\ \downarrow & & \downarrow \\ \mathscr{F}_x & \xrightarrow[\varphi_x]{} & \mathscr{G}_x \end{array}$$

we see that $s_x \in \mathscr{F}_x$ is mapped to 0, whence $s_x \in \ker(\varphi_x)$. Hence $(\ker \varphi)_x \subseteq \ker(\varphi_x)$. On the other hand, choose an element $t_x \in \ker(\varphi_x)$. Choose a pair $\langle V, t \rangle$ representing t_x. Then $\varphi_V(t) = u$, where $u_x = 0 \in \mathscr{G}_x$. Yet then there must exist some neighborhood $W \subseteq V$ of x such that $u|_W = 0$, whence $\varphi_W(t|_W) = u|_W = 0$. Hence $(t|_W)_x \in (\ker \varphi_W)_x$, whence $t_x \in (\ker \varphi)_x$. Therefore $(\ker \varphi)_x \supseteq \ker(\varphi)_x$, so we have equality.

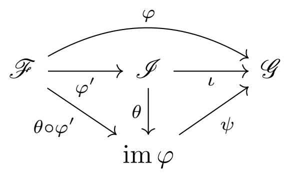
Now choose a point $x \in X$ and an element $s_x \in (\operatorname{im} \varphi)_x$. Choose a pair $\langle U, s \rangle$ representing s_x. Then there exists some $t \in \mathscr{F}(U)$ such that $\varphi_U(t) = s$, so in particular $\varphi_U(t)_x = s_x$. But then, using the commutative diagram above, this implies $\varphi(t_x) = s_x$ so $s_x \in \operatorname{im}(\varphi_x)$. Hence $(\operatorname{im} \varphi)_x \subseteq \operatorname{im}(\varphi_x)$. On the other hand, choose an element $s_x \in \operatorname{im}(\varphi_x)$. Then there exists $t_x \in \mathscr{F}_x$ with $\varphi_x(t_x) = s_x$. Choose a pair $\langle U, t \rangle$ representing t_x. Then $\varphi_U(t)$ satisfies $\varphi_U(t)_x = \varphi_x(t_x) = s_x$ using the commutative diagram above. Hence $s_x \in (\operatorname{im} \varphi)_x$, so $(\operatorname{im} \varphi)_x \supseteq \operatorname{im}(\varphi_x)$, and we have equality.

(2): Suppose φ is injective. Then $\ker(\varphi_x) = (\ker \varphi)_x = 0_x = 0$ for each $x \in X$, so each induced stalk map is injective. Conversely, suppose that $s \in \ker \varphi_U$. Then $s_x \in \ker \varphi_x$ for each $x \in U$, so $s_x = 0$ for each $x \in U$. But then by Lemma 2.9, $s = 0$. Hence $\ker \varphi_U = 0$ for each open $U \subseteq X$, so φ is injective.

Suppose φ is surjective. Then $\operatorname{im}(\varphi_x) = (\operatorname{im} \varphi)_x = \mathscr{G}_x$ for each $x \in X$, so each induced stalk map is surjective. On the other hand, suppose that $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves such that φ_x is surjective for each $x \in X$. Now, let $\mathscr{I}$ denote the presheaf image of φ. Then, there are natural morphisms $\varphi' : \mathscr{F} \to \mathscr{I}$ and $\iota : \mathscr{I} \to \mathscr{G}$ such that the following diagram commutes:

$$\mathscr{F} \xrightarrow[\varphi']{} \mathscr{I} \xrightarrow[\iota]{} \mathscr{G}, \qquad \varphi = \iota \circ \varphi' : \mathscr{F} \to \mathscr{G}$$

But then, by the universal property of sheafification, $\iota : \mathscr{I} \to \mathscr{G}$ factors through $\operatorname{im}\varphi \to \mathscr{G}$. Hence we have the following commutative diagram:



The goal is to demonstrate that ψ is an isomorphism, since then by definition φ is surjective. Yet notice that $(\operatorname{im}\varphi)_x = \operatorname{im}(\varphi_x) \simeq \mathscr{G}_x$ along ψ_x by assumption, so ψ is an isomorphism on stalks and therefore an isomorphism by Proposition 2.12. Therefore $\operatorname{im}\varphi \simeq \mathscr{G}$ along the natural map ψ, whence φ is surjective.

(3): The sequence $\cdots \to \mathscr{F}^{i-1} \overset{\varphi^{i-1}}{\to} \mathscr{F}^{i} \overset{\varphi^{i}}{\to} \mathscr{F}^{i+1} \to \cdots$ of sheaves is exact if and only if $\operatorname{im}\varphi^{i-1} = \ker\varphi^{i}$ for each i. Yet, by Lemma 2.9, this happens iff $(\operatorname{im}\varphi^{-1})_x = (\ker\varphi^{i})_x$ for each x and each i. By (1), this happens iff $\operatorname{im}\varphi_x^{i-1} = \ker\varphi_x^{i}$ for each x and each i. Yet this means precisely that $\cdots \to \mathscr{F}_x^{i-1} \overset{\varphi_x^{i-1}}{\to} \mathscr{F}_x^{i} \overset{\varphi_x^{i}}{\to} \mathscr{F}_x^{i+1} \to \cdots$ is exact for each x. □

Corollary 2.25.1 (A Sanity Check)**.** *A morphism of sheaves is an isomorphism if and only if it is injective and surjective.*

Proof. This immediately follows from Proposition 2.25 and Proposition 2.12. □

Next, let us discuss the following criterion for surjectivity:

Proposition 2.26 (Surjectivity Criterion)**.** *Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on X. Then φ is surjective if and only if the following condition holds: for every open set $U \subseteq X$, and for every $s \in \mathscr{G}(U)$, there is a covering $\{U_i\}$ of U, and there are elements $t_i \in \mathscr{F}(U_i)$, such that $\varphi(t_i) = s|_{U_i}$ for all i.*

Proof. Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on X. Then, if φ is surjective, φ_x is surjective for all $x \in X$ (by Proposition 2.25). Yet this means precisely that, for any open set $U \subseteq X$, any point $x \in U$, and any $s \in \mathscr{G}(U)$, there exist $\langle V(x), t(x)\rangle \in \mathscr{F}_x$ with $t(x) \in \mathscr{F}(V(x))$ and $V(x)$ an open neighborhood of x such that $\varphi_{V(x)}(t(x))_x = s_x$. By shrinking $V(x)$ if necessary, we may even choose $\langle V(x), t(x)\rangle \in \mathscr{F}_x$ to be such that $\varphi_{V(x)}(t(x)) = s|_{U(x)}$. Yet the collection $\{V(x)\}_{x\in U}$ covers U and satisfies the condition.

On the other hand, suppose that the condition is satisfied. To show that φ is surjective, it suffices to show that φ_x is surjective for an arbitrary $x \in X$. Now, choose an arbitrary element $s_x \in \mathscr{G}_x$ with representative $\langle U, s\rangle$. Now, by assumption, there exists an open covering $\{U_i\}$ of U and there are elements $t_i \in \mathscr{F}(U_i)$ such that $\varphi(t_i) = s|_{U_i}$ for all i. Now, x lies in U_i for some i, and $\varphi(t_i) = s|_{U_i}$ implies that $\varphi_x((t_i)_x) = \varphi(t_i)_x = s_x$, so φ_x is indeed surjective, and we are done. □

Let us also illustrate that what one might expect (that φ is surjective if and only if φ_U is surjective for each open set $U \subseteq X$), is actually false.

Proposition 2.27 (Surjectivity Counterexample)**.** *There exists a morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$ of X such that (i) φ is surjective (ii) φ_U is not surjective for some open $U \subseteq X$.*

Proof. Examples come from this StackExchange post. First, we begin with a classical example from complex analysis. Let $X = \mathbb{C} \setminus \{0\}$ be the punctured complex plane, $\mathscr{F}$ the sheaf of holomorphic functions, and $\mathscr{G}$ the sheaf of nowhere-zero holomorphic functions. Let $\varphi : \mathscr{F} \to \mathscr{G}$ send any holomorphic function f to $\exp(f)$. Then, at stalks, φ is surjective. This follows because we can take the logarithm of any nonvanishing function on any open disk not containing 0; in other words, we can take logarithms on nonvanishing functions on sufficiently small open sets. However, we cannot take logarithms on all open sets; for example, φ_X is not surjective since there is no holomorphic function $f : X \to \mathbb{C}$ such that $\exp f(z) = z$ for all non-zero z (recall that the logarithm cannot be defined on the punctured complex plane).

This example is extremely helpful and offers great intuition if one is familiar with complex analysis. However, if one is not familiar with complex analysis, or wants a counterexample with minimal effort, consider the following. Let $X = \mathbb{R}$. Define $\mathscr{F}$ to be the constant sheaf $\mathbb{Z}$; that is, $\mathscr{F}(U) = \mathbb{Z}$ for any open set $U \subseteq X$ and the restriction maps are just the identity. Similarly, define $\mathscr{G}$ as follows: $\mathscr{G}(U) = \mathbb{Z}^{|\{0,1\} \cap U|}$ (again, the restriction maps are obvious). Then, using the natural map $\mathbb{Z} \to \mathbb{Z}^k$ given by $1 \mapsto (1, \dots, 1)$, we define a morphism $\varphi : \mathscr{F} \to \mathscr{G}$. φ_x is either an isomorphism $\mathbb{Z} \to \mathbb{Z}$ (if $x = 0, 1$) or the trivial surjection $\mathbb{Z} \to 0$ (otherwise); in either case, φ_x is surjective so φ is surjective. However, $\varphi_X : \mathbb{Z} \to \mathbb{Z}^2$ is not surjective. □

Proposition 2.28 (Isomorphism Theorems)**.** *If* $\varphi : \mathscr{F} \to \mathscr{G}$ *is a morphism of sheaves, then* $\operatorname{im}\varphi \simeq \mathscr{F} / \ker\varphi$ *and* $\operatorname{coker}\varphi \simeq \mathscr{G} / \operatorname{im}\varphi$.

Proof. Consider the natural morphism from the presheaf $U \mapsto \mathscr{F}(U)/\ker\varphi_U$ to the presheaf $U \mapsto \operatorname{im}\varphi_U$. Compose this with the sheafification map from the presheaf image to the image sheaf $\operatorname{im}\varphi$, to get a map from the presheaf $U \mapsto \mathscr{F}(U)/\ker\varphi_U$ to the image sheaf. Then, by the universal property of sheafification, this gives a map $\mathscr{F} / \ker\varphi \to \operatorname{im}\varphi$. Since sheafification does not change stalks, we easily check that this map is an isomorphism on stalks and therefore an isomorphism by Proposition 2.12.

The second fact is proven identically and is thereby left as an exercise to the reader. □

Next, we will discuss exact sequences of sheaves.

Proposition 2.29 (Short Exact Sequences of Sheaves)**.** *Let* $\mathscr{F}'$ *be a subsheaf of a sheaf* $\mathscr{F}$*. Then the natural map* $\mathscr{F} \to \mathscr{F}/\mathscr{F}'$ *is surjective with kernel* $\mathscr{F}'$*. That is, just like in the case of abelian groups, there is an exact sequence* $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}/\mathscr{F}' \to 0$.

Just like in the case of abelian groups, the converse is also true: if $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ *is an exact sequence, show that* $\mathscr{F}'$ *is isomorphic to a subsheaf of* $\mathscr{F}$*, and that* $\mathscr{F}''$ *is isomorphic to the quotient of* $\mathscr{F}$ *by this subsheaf.*

Proof. Let $\mathscr{F}'$ be a subsheaf of $\mathscr{F}$. Then the natural map $\mathscr{F} \to \mathscr{F}/\mathscr{F}'$ is given by composing the natural surjection $\mathscr{F}(U) \twoheadrightarrow \mathscr{F}(U)/\mathscr{F}'(U)$ with the sheafification map $\theta_U : \mathscr{F}(U)/\mathscr{F}'(U) \to (\mathscr{F}/\mathscr{F}')(U)$. Similarly, there is a natural map $\mathscr{F}' \hookrightarrow \mathscr{F}$ since $\mathscr{F}'$ is a subsheaf of $\mathscr{F}$. Hence we have a sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}/\mathscr{F}' \to 0$; it suffices to check that this sequence is exact. Now, recall that because the sheafification map is an isomorphism on stalks, $(\mathscr{F}/\mathscr{F}')_x \simeq \mathscr{F}_x/\mathscr{F}'_x$ for any $x \in X$. But then $0 \to \mathscr{F}'_x \to \mathscr{F}_x \to \mathscr{F}_x/\mathscr{F}'_x \to 0$ is obviously exact, so by Proposition 2.25, so is $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}/\mathscr{F}' \to 0$. This implies, in particular, that $\mathscr{F} \to \mathscr{F}/\mathscr{F}'$ is surjective and has kernel $\mathscr{F}'$.

On the other hand, suppose $0 \to \mathscr{F}' \xrightarrow{\varphi} \mathscr{F} \xrightarrow{\psi} \mathscr{F}'' \to 0$. Then, $\varphi : \mathscr{F}' \to \mathscr{F}$ is injective, so $\mathscr{F}' \simeq \mathscr{F}'/0 \simeq \operatorname{im}\varphi$ (by, say Proposition 2.28), which is a subsheaf of $\mathscr{F}$. Then, by exactness, ψ is surjective; that is, $\operatorname{im}\psi \simeq \mathscr{F}''$. Hence by Proposition 2.28, $\mathscr{F}'' \simeq \mathscr{F} / \ker\psi$. But by exactness $\ker\psi = \operatorname{im}\varphi$, so $\mathscr{F}'' \simeq \mathscr{F} / \operatorname{im}\varphi$, which is exactly what we are asked to show. Therefore we are done. □

Theorem 2.30 (Exactness of Evaluation and Flasque Sheaves)**.**

1. *Suppose that* $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}''$ *is an exact sequence of sheaves of abelian groups on* X*. Then, for any open* $U \subseteq X$*,* $0 \to \mathscr{F}'(U) \to \mathscr{F}(U) \to \mathscr{F}''(U)$ *is exact; in other words, evaluation at* U *is a left-exact functor. In general, because not all surjective maps have surjective component maps, evaluation at* U *is not an exact functor.*

2. *Suppose that* $\mathscr{F}$ *is a sheaf such that any restriction map is surjective. Then* $\mathscr{F}$ *is called flasque. If* $\mathscr{F}'$ *is flasque, and* $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ *is an exact sequence, then for any open sest* $U \subseteq X$*, the sequence* $0 \to \mathscr{F}'(U) \to \mathscr{F}(U) \to \mathscr{F}''(U) \to 0$ *of abelian groups is also exact.*

3. *If* $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ *is an exact sequence of sheaves, and if* $\mathscr{F}'$ *and* $\mathscr{F}$ *are flasque, then* $\mathscr{F}''$ *is flasque.*

Proof.
(1): Firstly, notice that exactness at $\mathscr{F}(U)$ follows immediately, because exactness at $\mathscr{F}$ implies that $\mathscr{F}' \to \mathscr{F}$ is injective which implies that $\mathscr{F}'(U) \to \mathscr{F}(U)$ is injective for each open set $U \subseteq X$. Therefore, it suffices to show exactness at $\mathscr{F}'(U)$. Let φ denote the map $\mathscr{F}' \to \mathscr{F}$ and ψ denote the map $\mathscr{F} \to \mathscr{F}''$. Now choose

$s \in \mathscr{F}'(U)$. Then we have $(\phi_U(\psi_U(s)))_x = \phi_x(\psi_x(s_x)) = 0$ (the last equality follows from the exactness of taking stalks, see Proposition 2.25). Therefore $\varphi(s) \in \ker \psi_U$, so $\operatorname{im} \varphi_U \subseteq \ker \psi_U$.

On the other hand, take $s \in \ker \psi_U$. Let x be a point in U. Since taking stalks in exact, the $s_x \in \ker \psi_x$ lies in the image of $\operatorname{im} \varphi_x$. That is, there exists $t_x \in \mathscr{F}'_x$ such that $s_x = \varphi_x(t_x)$. Now, let t_x be represented by $\langle V(x), t(x) \rangle$. Shrinking $V(x)$ if neecssary, we may suppose that $\varphi_{V(x)}(t(x)) = s|_{V(x)}$. Now, I claim that we may glue together the $t(x)$ to a section $t \in \mathscr{F}'(U)$. For this, first notice that the collection $\{V(x)\}_{x \in U}$ covers U. Secondly, notice that if $x, y \in U$, then

$$\varphi_{V(x)\cap V(y)}(t(x)|_{V(x)\cap V(y)}) = s|_{V(x)\cap V(y)} = \varphi_{V(x)\cap V(y)}(t(y)|_{V(x)\cap V(y)})$$

which by injectivity of φ implies that $t(x)|_{V(x)\cap V(y)} = t(y)|_{V(x)\cap V(y)}$. Hence the sections are compatible, so we may indeed glue them to a section $t \in \mathscr{F}'(U)$. Then it is easy to check that $\varphi_U(t) = s$ (since we have equality on the cover $\{V(x)\}_{x\in U}$ and may apply uniqueness). Hence $\ker \psi_U \subseteq \operatorname{im} \varphi_U$, so we have equality.

(2): Take an open set $U \subseteq X$. By part (1), it suffices to show that $\mathscr{F}(U) \to \mathscr{F}''(U)$ is surjective. Let $s \in \mathscr{F}''(U)$. Since $\mathscr{F} \to \mathscr{F}''$ is surjective, by Proposition 2.26, there exists an open cover $\{U_i\}_{i\in I}$ of U and sections $t_i \in \mathscr{F}(U_i)$ with $t_i \mapsto s|_{U_i}$. We will use Zorn's Lemma to find the "biggest possible" section obtained by gluing the t_i together, and show that in fact this section lies in $\mathscr{F}(U)$ and maps to s.

Let $\mathcal{S}$ be the set of pairs (J, z), where J is a subset of the index set I, and $z \in \mathscr{F}(\bigcup_{j\in J} U_j)$ satisfies $z \mapsto s|_{\bigcup_{j\in J} U_j}$. Place the natural partial ordering on $\mathcal{S}$; $(J, z) \leq (J', z')$ if $J \subseteq J'$ and z' restricts to z. The set $\mathcal{S}$ is clearly nonempty, and any chain of S is bounded above by the sheaf axiom, so by Zorn's Lemm $\mathcal{S}$ has a maximal element (I', z). Now, we will show that $I = I'$, so $z \in \mathscr{F}(U)$.

Suppose that $I' \neq I$. Then, there exists $i \in I \setminus I'$. Set $V = \bigcup_{j\in I'} U_j$ and let $t_i \in \mathscr{F}(U_i)$ be the element described earlier. Now, define $x = z|_{V\cap U_i} - t_i|_{V\cap U_i}$. Notice that $x \mapsto 0 \in \mathscr{F}''(V \cap U_i)$, so there exists $v_i \in \mathscr{F}'(V \cap U_i)$ mapping to x. Since $\mathscr{F}'$ is flasque, we may lift v_i to $w_i \in \mathscr{F}'(U_i)$ and define $t'_i = t_i + w_i$. Then z, t'_i are compatible sections and glue to $z' \in \mathscr{F}(V \cup U_i)$. Clearly $z' \mapsto s|_{V\cup U_i}$. Therefore, $(I, z) < (I', z')$. Since I' was chosen to be maximal, this is a contradiction, so the assumption $I \neq I'$ was wrong.

Hence $z \in \mathscr{F}(U)$ and by construction of $\mathcal{S}$, $z \mapsto s|_U = s$, as desired. Therefore we are done.

(3): This is simple. Suppose $V \subseteq U$. Then, we have a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{F}'(U) & \longrightarrow & \mathscr{F}(U) & \longrightarrow & \mathscr{F}''(U) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{F}'(V) & \longrightarrow & \mathscr{F}(V) & \longrightarrow & \mathscr{F}''(V) & \longrightarrow & 0
\end{array}$$

Since $\mathscr{F}'$ is flasque, $\mathscr{F}(V) \to \mathscr{F}''(V)$ is surjective by (2). Since $\mathscr{F}$ is flasque, $\mathscr{F}(U) \to \mathscr{F}(V)$ is surjective by definition. Therefore, the composition $\mathscr{F}(U) \to \mathscr{F}(V) \to \mathscr{F}''(V)$ is surjective, so by commutativity the composition $\mathscr{F}(U) \to \mathscr{F}''(U) \to \mathscr{F}''(V)$ must also be surjective. But then, in particular, $\mathscr{F}''(U) \to \mathscr{F}''(V)$ is surjective, so $\mathscr{F}''$ is also flasque. In particular, the image of any flasque sheaf is flasque. ☐

Finally, we will conclude with a discussion of ways to *create* sheaves from incomplete data via either extending or gluing. These techniques can (and will) be extremely helpful in simplifying later definitions.

Theorem 2.31 (Extending Sheaves on a Base)**.** *Suppose that X is a topological space, and $\mathscr{B} = \{B_i\}$ is a base for the topology on X. Then suppose we have an "incomplete sheaf", called a* sheaf on the base *$\mathscr{B}$, which assigns to each B_i an abelian group $F(B_i)$ and to each inclusion $B_i \subseteq B_j$ a restriction map $\operatorname{res}_{B_j,B_i} : F(B_j) \to F(B_i)$ such that if $B_i \subseteq B_j \subseteq B_k$, $\operatorname{res}_{B_k,B_i} = \operatorname{res}_{B_j,B_i} \circ \operatorname{res}_{B_k \circ B_j}$.*

Suppose further that F satisfies the following axioms:

(1) If $B \in \mathscr{B}$ has an open cover $\{B_j\} \subseteq \mathscr{B}$ and $s \in F(B)$ is such that $f|_{B_j} = 0$ for each j, then $s = 0$.

(2) If $B \in \mathscr{B}$ has an open cover $\{B_j\} \subseteq \mathscr{B}$, and we have $s_j \in F(B_j)$ such that $s_j|_{B_l} = s_k|_{B_l}$ for any $B_l \subseteq B_j \cap B_k$, there exists $s \in F(B)$ such that $s|_{B_j} = s_j$ for each j.

Then there is a sheaf $\mathscr{F}$, unique up to unique isomorphism, extending F (that is, with isomorphisms $\mathscr{F}(B_i) \simeq F(B_i)$ agreeing with the restriction maps of F).

Proof. The construction offered here is from Vakil's *Foundations of Algebraic Geometry*, 2.5. The key is to define $\mathscr{F}$ as the sheaf of "compatible germs of F". Namely, define the *stalk* of a sheaf on the base F at $x \in X$ as $F_x = \varinjlim F(B_i)$, where the direct limit is taken over all B_i containing x. One may also consider the explicit construction using pairs $\langle B_i, s\rangle$ analogous to the explicit construction for stalks of sheaves (see Definition 2.7).

Define $\mathscr{F}$ as follows:

$$\mathscr{F}(U) := \{(f_x \in F_x)_{x\in U} \mid \text{for all } x \in U, \text{ there is } B \in \mathscr{B} \text{ with } x \in B \subseteq U,\ s \in F(B),\ s_q = f_q \text{ for all } q \in B\}$$

Also give $\mathscr{F}$ the natural restriction maps, and notice that $\mathscr{F}(U)$ has a natural abelian group structure. Therefore $\mathscr{F}$ is a presheaf. To see that it is a sheaf is similarly simple. Finally, one may verify that the natural map $F(B_i) \to \mathscr{F}(B_i)$ given by sending $s \in F(B_i)$ to $(s_x \in F_x)_{x\in B_i}$ is an isomorphism. □

Theorem 2.32 (Extending Morphisms on a Base)**.** *Suppose X is a topological space, and $\mathscr{B} = \{B_i\}$ is a base for the topology on X. Then a* morphism $\varphi : F \to G$ of sheaves on the base $\mathscr{B}$ *is a collection of maps $\varphi_{B_i} : F(B_i) \to G(B_i)$ such that, for any inclusion $B_i \subseteq B_j$, the following diagram commutes:*

$$\begin{array}{ccc} F(B_i) & \xrightarrow{\varphi_{B_i}} & G(B_i) \\ {\scriptstyle \mathrm{res}_{B_i,B_j}}\downarrow & & \downarrow{\scriptstyle \mathrm{res}_{B_i,B_j}} \\ F(B_j) & \xrightarrow[\varphi_{B_i}]{} & G(B_j) \end{array}$$

Recall from the previous theorem that F and G induce (unique up to unique isomorphism) sheaves $\mathscr{F}$ and $\mathscr{G}$ extending F and G. Similarly, φ induces a unique morphism of sheaves $\mathscr{F} \to \mathscr{G}$ extending φ.

Proof. The proof follows from applying the definition of the extended sheaf in Theorem 2.31. □

Theorem 2.33 (Gluing Sheaves)**.** *Let X be a topological space with open cover $\{U_i\}$. Suppose that we are given for each i a sheaf $\mathscr{F}_i$ on U_i, and for each i, j an isomorphism $\varphi_{ij} : \mathscr{F}_i|_{U_i\cap U_j} \xrightarrow{\sim} \mathscr{F}_j|_{U_i\cap U_j}$ such that (1) for each i, $\varphi_{ii} = \mathrm{id}$, and (2) for each i, j, k, $\varphi_{ij} = \varphi_{jk} \circ \varphi_{ij}$ on $U_i \cap U_j \cap U_k$. Then there exists a unique sheaf $\mathscr{F}$ on X, together with isomorphisms $\psi_i : \mathscr{F}|_{U_i} \xrightarrow{\sim} \mathscr{F}_i$ such that for each i, j, $\psi_j = \varphi_{ij} \circ \psi_i$ on $U_i \cap U_j$. We say that $\mathscr{F}$ is obtained by* gluing *the sheaves $\mathscr{F}_i$ along the isomorphisms φ_{ij}.*

Proof. This follows from Theorem 2.31. To see why, let $\{U_i\}$ be an open cover of X and let $\mathscr{B}$ be the collection of all open sets contained in one of the U_i. Then it is easy to see that $\mathscr{B}$ is a base for the topology of X. Furthermore, the data provided allows us to define a (unique up to isomorphism) sheaf F on the base $\mathscr{B}$, which by Theorem 2.31 we may uniquely extend to get the desired sheaf $\mathscr{F}$ on X. □

Theorem 2.34 (Gluing Morphisms of Sheaves)**.** *Let X be a topological space with open cover $\{U_i\}$. Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves on X. Suppose that we are given, for each i, a morphism $\varphi_i : \mathscr{F}|_{U_i} \to \mathscr{G}|_{U_i}$, and suppose that furthermore these morphisms are compatible in the sense for any U_i, U_j, the restriction $\varphi_i|_{U_i\cap U_j}$ is isomorphic to the restriction $\varphi_j|_{U_i\cap U_j}$. Then there exists a unique morphism $\varphi : \mathscr{F} \to \mathscr{G}$ together with isomorphisms $\varphi|_{U_i} \xrightarrow{\sim} \varphi_i$. We say that φ is obtained by* gluing *the morphisms φ_i.*

Proof. As the above theorem follows from Theorem 2.31, this follows from Theorem 2.32. □

2.2 Ringed and Locally Ringed Spaces

Definition 2.35 (Ringed Space)**.** A *ringed space* is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on X. X is called the *underlying space* of the ringed space, and $\mathcal{O}_X$ is called the structure sheaf; however, by abuse of notation, we obtain denote such a ringed space just by X. To make sure this abuse of notation does not cause confusion, we often denote the underlying space of a ringed space $(X, \mathcal{O}_X)$ (which, again, we sometimes denote by just X) by $\mathrm{sp}(X)$.

Definition 2.36 (Morphism of Ringed Spaces)**.** A *morphism of ringed spaces* from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair $(f, f^\#)$ of a continuous map $f : X \to Y$ and a map $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$. By abuse of notation, we sometimes denote the pair $(f, f^\#)$ by just f, but it is not determined by f.

Definition 2.37 (Locally Ringed Space)**.** A *locally ringed space* is a ringed space such that the stalk $\mathcal{O}_{X,x}$ is a local ring for each $x \in X$.

Definition 2.38 (Local Ring Homomorphism)**.** If $(A, \mathfrak{m}_A)$ and $(B, \mathfrak{m}_B)$ are local rings, then a homomorphism $\varphi : A \to B$ is called a *local homomorphism* if $\varphi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$.

Definition 2.39 (Morphism of Locally Ringed Spaces)**.** A *morphism of locally ringed spaces* is a morphism of ringed spaces $(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ such that for each point $x \in X$, the induced map of local rings $f_x^\# : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local ring homomorphism.

In more detail, notice that the morphism of sheaves $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ induces a homomorphism of rings $\mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V))$ for every point set V in Y. As V ranges over all neighborhoods of $f(x)$, $f^{-1}(V)$ ranges over a subset of the neighborhoods of x. Hence we obtain a map $\mathcal{O}_{Y,f(x)} = \varinjlim_V \mathcal{O}_Y(V) \to \varinjlim_V \mathcal{O}_X(f^{-1}(V)) = \mathcal{O}_{X,x}$, which is the described "induced map".

Definition 2.40 ($\operatorname{Spec} A$)**.** Let A be a commutative ring. Then $\operatorname{Spec} A$, the *spectrum of* A, is the set of all prime ideals of A. For any ideal $\mathfrak{a} \lhd A$, $V(\mathfrak{a})$ is defined to be the set of all prime ideals which contain $\mathfrak{a}$. Since $V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})$, $\bigcap V(\mathfrak{a}_i) = V(\sum \mathfrak{a}_i)$, $V(A) = \varnothing$, and $V(0) = \operatorname{Spec} A$, subsets of $\operatorname{Spec} A$ of the form $V(\mathfrak{a})$ satisfy the axioms of closed sets for a topological space. Therefore, we may place a topology on $\operatorname{Spec} A$ by letting sets of the form $V(\mathfrak{a})$ be the closed sets.

Definition 2.41 (Basic Affine Open)**.** If A is a commutative ring, and $f \in A$, then $D(f)$ denotes the open complement of $V((f))$, and is called a *basic affine open.*

Proposition 2.42. *If A is a commutative ring, then $\{D(f)\}_{f \in A}$ is a base for the topology of* $\operatorname{Spec} A$.

Proof. Suppose U is an open neighborhood of a point $\mathfrak{p}$ in $\operatorname{Spec} A$. Then $U = \operatorname{Spec} A \setminus V(\mathfrak{a})$ for some ideal $\mathfrak{a} \lhd A$. Then, $\mathfrak{p} \notin V(\mathfrak{a})$ implies $\mathfrak{p} \not\supseteq \mathfrak{a}$, so there is an $f \in \mathfrak{a}$ such that $f \notin \mathfrak{p}$. But then $\mathfrak{p} \in D(f)$ and $D(f) \cap V(\mathfrak{a}) = \varnothing$, whence $D(f) \subseteq U$, as desired. □

Definition 2.43 (Spectrum of a Ring)**.** Let A be a commutative ring. Then the *spectrum of* A is the ringed space $(\operatorname{Spec} A, \mathcal{O})$, where the structure sheaf $\mathcal{O}$ is defined as follows:

1. For an open set $U \subseteq \operatorname{Spec} A$, define $\mathcal{O}(U)$ to be the set of functions $s : U \to \coprod_{\mathfrak{p} \in U} A_\mathfrak{p}$ such that $s(\mathfrak{p}) \in A_\mathfrak{p}$ for each $\mathfrak{p}$ and such that s is locally a quotient of elements of A. Precisely, we require that for each $\mathfrak{p} \in U$, there is a neighborhood V of $\mathfrak{p}$ contained in U and elements $a, f \in A$ such that for each $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = a/f$ in $A_\mathfrak{q}$.

2. The restriction map $\mathcal{O}(U) \to \mathcal{O}(V)$ is the natural restriction of functions.

It is clear that $\mathcal{O}$ is a presheaf and indeed a sheaf, so $(\operatorname{Spec} A, \mathcal{O})$ is indeed a ringed space.

Proposition 2.44 (Stalks and Sections of Spectra)**.** *Let A be a ring and* $(\operatorname{Spec} A, \mathcal{O})$ *its spectrum.*

(a) For any $\mathfrak{p} \in \operatorname{Spec} A$, the stalk $\mathcal{O}_\mathfrak{p}$ of the sheaf $\mathcal{O}$ is isomorphic to the local ring $A_\mathfrak{p}$. In particular, $(\operatorname{Spec} A, \mathcal{O})$ *is a* locally *ringed space.*

(b) For any element $f \in A$, the ring $\mathcal{O}(D(f))$ is isomorphic to the localized ring A_f. In particular, the ring of global sections $\mathcal{O}(\operatorname{Spec} A) \simeq A$.

Proof.
(a): Define a map $\varphi : \mathcal{O}_\mathfrak{p} \to A_\mathfrak{p}$ by sending any local section s in a neighborhood of $\mathfrak{p}$ to its value $s(\mathfrak{p}) \in A_\mathfrak{p}$. Plainly, this is a homomorphism. To see that it is surjective, choose an element $a/f = A_\mathfrak{p}$. Then $D(f)$ is an open neighborhood of $\mathfrak{p}$, and the constant function a/f is a section of $\mathcal{O}(D(f))$. Yet $\langle \mathcal{O}(D(f)), a/f \rangle \in \mathcal{O}_\mathfrak{p}$ is sent to a/f by φ. Therefore, it suffices to show that φ is injective.

Let $\langle U, s\rangle$ and $\langle V, t\rangle$ be two elements of $\mathcal{O}_\mathfrak{p}$ such that $s(\mathfrak{p}) = t(\mathfrak{p})$ in $A_\mathfrak{p}$ (that is, they have the same image under φ). Then, there exists a neighborhood $W_1 \subseteq U$ of $\mathfrak{p}$ such that $s = a/f$, and similarly there exists a neighborhood $W_2 \subseteq V$ of $\mathfrak{p}$ such that $t = b/g$, where $f, g \notin \mathfrak{p}$. Now let $W = W_1 \cap W_2$, so that in W, $s = a/f$ *and* $t = b/g$. Since these two elements have the same image in $A_\mathfrak{p}$, it follows that there is an $h \notin \mathfrak{p}$ such that $h(ga - fb) = 0$. Therefore $a/f = b/g$ in every local ring $A_\mathfrak{q}$ such that $f, g, h \notin \mathfrak{q}$. But the set of such $\mathfrak{q}$ is the open set $D(f) \cap D(g) \cap D(h)$, which is a neighborhood of $\mathfrak{p}$. Hence $\langle U, s\rangle$ and $\langle V, t\rangle$ are equal in a neighborhood of $\mathfrak{p}$, so they are equal in $\mathcal{O}_\mathfrak{p}$ and φ is injective.

(b): For each $\mathfrak{p}$ not containing f, there is a natural map $\iota_\mathfrak{p} : A_f \to A_\mathfrak{p}$. Let $\psi : A_f \to \mathcal{O}(D(f))$ be the morphism given by sending a/f^n to the section $s \in \mathcal{O}(D(f))$ which maps $\mathfrak{p}$ to $\iota_\mathfrak{p}(a/f^n)$. Proving that this is injective and surjective is laborious but straightforward, so it either left as an exercise to the reader or can be read on pg. 70-71 of Hartshorne. Finally, the particular statement follows from letting $f = 1$, since $D(1) = \operatorname{Spec}(A)$. ☐

Proposition 2.45 (Morphisms of Ring Spectra)**.** *Let A be a ring and $(\operatorname{Spec} A, \mathcal{O})$ its spectrum. Then if $\varphi : A \to B$ is a homomorphism of rings, φ induces a natural morphism of locally ringed spaces $(f, f^\#) : (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \to (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$. Furthermore, if A and B are rings, then any morphism of locally ringed spaces from $\operatorname{Spec} B$ to $\operatorname{Spec} B$ is induced by a homomorphism of rings $\varphi : A \to B$.*

Proof. Suppose that $\varphi : A \to B$ is a homomorphism of rings. Then define a map $f : \operatorname{Spec} B \to \operatorname{Spec} A$ by $f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$. This is continuous, because the preimage of a closed set $V(\mathfrak{a})$ is the closed set $V(\varphi(\mathfrak{a}))$. Now, for any $\mathfrak{p} \in \operatorname{Spec} B$, φ induces a local ring homomorphism $\varphi_\mathfrak{p} : A_{\varphi^{-1}(\mathfrak{p})} \to B_\mathfrak{p}$. Therefore, we may define $f^\# : \mathcal{O}_{\operatorname{Spec} A}(V) \to \mathcal{O}_{\operatorname{Spec} B}(f^{-1}(V))$ by sending a section s by composing f on the right and the disjoint union of the $\varphi_\mathfrak{p}$ on the left. This is a morphism of ringed spaces, and furthermore a morphism of locally ringed spaces because the induced maps on the stalks are just the local ring homomorphisms $\varphi_\mathfrak{p}$.

Conversely, suppose that we are given a morphism of locally ringed spaces $(f, f^\#)$ from $\operatorname{Spec} B$ to $\operatorname{Spec} A$. Then by taking global sections, $f^\#$ induces a homomorphism of rings $\varphi : A \to B$. For any $\mathfrak{p} \in \operatorname{Spec} B$, we have an induced local homomorphism $A_{f(\mathfrak{p})} \to B_\mathfrak{p}$ on the stalks, such that the below diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \downarrow & & \downarrow \\ A_{f(\mathfrak{p})} & \xrightarrow[f^\#_\mathfrak{p}]{} & B_\mathfrak{p} \end{array}$$

Since $f^\#$ is a local homomorphism, it follows that $\varphi^{-1}(\mathfrak{p}) = f(\mathfrak{p})$, so f coincides with the map $\operatorname{Spec} B \to \operatorname{Spec} A$ induced by φ. It is immediate that $f^\#$ is also induced by φ, so we are done. ☐

Lemma 2.46 (Basic Affine Opens are Spectra)**.** *Suppose that A is a commutative ring with spectra $\operatorname{Spec} A$ and $f \in A$ is an element. Then $D(f) \simeq \operatorname{Spec} A_f$ as locally ringed spaces.*

Proof. Firstly, notice that $D(f) = \{\mathfrak{p} \subseteq A \mid f \notin \mathfrak{p}\}$ is in a natural one-to-one correspondence with $\operatorname{Spec} A_f = \{\mathfrak{p} \in A \mid \mathfrak{p} \cap f = \varnothing\}$, and that furthermore this correspondence is continuous in both directions (because it is order-preserving). Therefore $D(f) \cong \operatorname{Spec} A_f$ as topological spaces. Yet furthermore, by Proposition 2.44, $\mathcal{O}_X(D(f)) = A_f$, so $\mathcal{O}_X|_{D(f)} \simeq \mathcal{O}_{A_f}$. Hence the result follows. ☐

Definition 2.47 (Evaluation of a Section at a Point)**.** Suppose that $(X, \mathcal{O}_X)$ is a locally ringed space, and that $x \in X$ is a point. Then, for any open set U containing x, we have a ring map $\mathcal{O}_X(U) \to \mathcal{O}_{X,x} \twoheadrightarrow k(x)$, the residue field of $\mathcal{O}_{X,x}$. The image of $s \in \mathcal{O}_X$ is denoted $s(x)$ and is called the *value of s at x*.

Example 2.48 (Evaluation is Evaluation of Functions)**.** For manifolds and algebraic sets, this recovers the usual notion of the value of a function at a point x. Check this for yourself with an example.

2.3 Schemes and Morphisms

Definition 2.49 (Affine Scheme)**.** An *affine scheme* is a locally ringed space $(X, \mathcal{O}_X)$ which is isomorphic (as a locally ringed space) to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ for some commutative ring A.

Definition 2.50 (Scheme)**.** A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ in which every point has an open neighborhood U such that the topological space U together with the restricted sheaf $\mathcal{O}_X|_U$ is an affine scheme. A *(iso)morphism of schemes* is a(n) (iso)morphism of locally ringed spaces between two schemes.

Following are a few basic results which we will use later. Firstly, we know that any scheme is covered by open affine schemes. However, something stronger is in fact true:

Lemma 2.51 (Affine Opens Form a Base)**.** *Suppose that $(X, \mathcal{O}_X)$ is a scheme, $x \in X$ is a point, and $U \subseteq X$ is a neighborhood of x. Then there exists a neighborhood $V \subseteq U$ of x such that $(V, \mathcal{O}_X|_V)$ is an affine scheme.*

Proof. Let V be an affine neighborhood of x, and suppose $V \simeq \operatorname{Spec} A$. Now, $V \cap U$ is an open set in V – indeed it is a neighborhood of x in U. Therefore, because the basic affine opens form a basis for the topology (see Proposition 2.42), there exists some $f \in A$ such that $D(f) \subseteq V \cap U$ is a neighborhood of x in V. Yet $D(f)$ is open in X as an open subset of the open subset $V \subseteq X$, and furthermore $D(f)$ is affine, isomorphic to $\operatorname{Spec} A_f$, by Lemma 2.46. Finally, $x \in D(f) \subseteq U$ by construction, so we are done. ☐

From this we may conclude that any open subset of a scheme is itself naturally a scheme.

Lemma 2.52 (Open Subscheme)**.** *Let $(X, \mathcal{O}_X)$ be a scheme and $U \subseteq X$ be an open subset. Then $(U, \mathcal{O}_X|_U)$ is a scheme, called an* open subscheme *of X.*

Definition 2.53 (Dimension and Codimension)**.** The *dimension* of a scheme X, denoted $\dim X$, is its dimension as a topological space. If X is an irreducible closed subset of X, then the *codimension* of Z in X, denoted $\operatorname{codim}(Z, X)$, is the supremum of integers n such that there exists a chain $Z = Z_0 \subseteq Z_1 \subseteq \cdots \subseteq Z_n$ of distinct closed irreducible subsets of X. If Y is any closed subset of X, we define

$$\operatorname{codim}(Y, X) = \inf_{\substack{Z \subseteq Y \\ \text{irreducible}}} \operatorname{codim}(Z, X).$$

Finally, it can be helpful to remember the following fact:

Lemma 2.54. *Suppose that $(X, \mathcal{O}_X)$ is a scheme and $U \subseteq X$ is a nonempty open set. Then $\mathcal{O}_X(U) \neq 0$.*

Proof. Choose a point $x \in U$. By Lemma 2.51, there exists a affine open $V \subseteq U$ containing x. Suppose $V \simeq \operatorname{Spec} A$. Yet since V is nonempty, A must be nonzero, so $\mathcal{O}_X(V) \simeq A$ is nonzero. However, there is a restriction map $\operatorname{res}_{U,V} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$. Since $\mathcal{O}_X(V)$ is nonzero, $0 \neq 1$ in $\mathcal{O}_X(V)$; furthermore, since it is a ring homomorphism, $\operatorname{res}_{U,V}(1) = 1$ and $\operatorname{res}_{U,V}(0) = 0$. Therefore $0 \neq 1 \in \mathcal{O}_X(U)$, whence $\mathcal{O}_X(U) \neq 0$. ☐

Definition 2.55 (Types of Schemes)**.** Let X be a scheme. Then,

(1) X is called *connected* if $\operatorname{sp}(X)$ is connected.

(2) X is called *irreducible* if $\operatorname{sp}(X)$ is irreducible.

(3) X is called *reduced* if $\mathcal{O}_X(U)$ is reduced for every open set U.

(4) X is called *integral* if $\mathcal{O}_X(U)$ is an integral domain for every open set U.

Proposition 2.56 (Reduced iff Stalks are Reduced)**.** *A scheme X is reduced if and only if $\mathcal{O}_{X,x}$ is reduced for each $x \in X$.*

Proof. Suppose that X is reduced; that is, the nilradical $\mathcal{N}(\mathcal{O}_X(U))$ of $\mathcal{O}_X(U)$ is zero for any open set $U \subseteq X$. Now, choose a point $x \in X$, and take an open affine neighborhood U of x. Then $U \simeq \operatorname{Spec} A$, and x corresponds to some prime ideal $\mathfrak{p}$. Then, because localization commutes with radicals, $\mathcal{N}(\mathcal{O}_{X,x}) = \mathcal{N}(A_\mathfrak{p}) = \mathcal{N}(A)_\mathfrak{p} = 0_\mathfrak{p} = 0$. Therefore $\mathcal{O}_{X,x}$ is also reduced for each $x \in X$.

Conversely, let $\mathcal{N}(\mathcal{O}_{X,x}) = 0$ for all $x \in X$. For any open $U \subseteq X$, pick a section $s \in \mathcal{O}_X(U)$ and assume that $s^n = 0$ for some n. Then, $s_x^n = 0$ for all $x \in U$, so by assumption $s_x = 0$ for all $x \in U$. Yet this implies that $s = 0$ by Lemma 2.9. Hence $\mathcal{O}_X(U)$ is a reduced ring, so indeed X is reduced. ☐

Now, obviously an affine scheme is integral if and only if it is both reduced and irreducible. As one might expect, the same is true for general schemes. However, this is not trivial to prove, and requires a short technical lemma (which is also a broadly useful fact).

Lemma 2.57. *Let X be a scheme. Take a section $f \in \mathcal{O}_X(U)$, and define U_f to be the subset of points $x \in U$ such that the stalks f_x of x is not contained in the maximal ideal $\mathfrak{m}_x$ of the local ring $\mathcal{O}_x$. Then U_f is an open subset of U.*

Proof. Since openness and the property of f_x not being contained in the maximal ideal $\mathfrak{m}_x$ of $\mathcal{O}_x$ are both local properties, we may assume that $U = X$. Furthermore, we may reduce to the affine case by taking an open affine cover $\{V_i\}$ of X. Therefore, assume that X is an affine scheme, isomorphic to $\operatorname{Spec} A$. The goal has been reduced to showing that if $f \in A$, then U_f is an open subset of U. For this, I claim that $U_f = D(f)$. Yet this is obvious, because $\mathfrak{p} \in D(f)$ iff $f \notin \mathfrak{p}$ iff $\frac{f}{1} \notin \mathfrak{p}_\mathfrak{p}$ iff $f \in U_f$. □

Notice the general strategy of reducing to the affine case, which is extremely useful.

Proposition 2.58. *A scheme is integral if and only if it is both reduced and irreducible.*

Proof. Suppose $(X, \mathcal{O}_X)$ is an integral scheme. Then by definition it is reduced. Furthermore, if X is not irreducible, then by Proposition 6.13, there exist two nonempty disjoint open subsets U_1 and U_2. But then $\mathcal{O}_X(U_1 \sqcup U_2) = \mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2)$ is not an integral domain, since nonemptiness implies $\mathcal{O}_X(U_1), \mathcal{O}_X(U_2) \neq 0$ (see Lemma 2.54). Hence by contraposition, integral implies irreducible.

Conversely, suppose that X is reduced and irreducible. Let $U \subseteq X$ be an open subset, and suppose that there are elements $f, g \in \mathcal{O}_X(U)$ with $fg = 0$. Let $Y = \{x \in U \mid f_x \in \mathfrak{m}_x\}$, and let $Z = \{x \in U \mid g_x \in \mathfrak{m}_x\}$. Then Y and Z are closed subsets of U by Lemma 2.57, and $Y \cap Z = U$. But X is irreducible, so U is irreducible (see Proposition 6.14), so one of Y or Z is equal to U, say $Y = U$. Then, given any open affine subset V of U, $D(f|_V) \subseteq V$ is empty. But this is equivalent to stating that $f|_V$ is nilpotent, which since X is reduced implies that $f|_V = 0$. Since open affine subsets cover U, by the uniqueness axiom this implies $f = 0$. If $Z = U$, then similarly $g = 0$. In either case, $fg = 0$ implies $f = 0$ or $g = 0$, or $\mathcal{O}_X(U)$ is an integral domain and we are done. □

Theorem 2.59. *Let A be a ring and let $(X, \mathcal{O}_X)$ be a scheme. Given a morphism $f : X \to \operatorname{Spec} A$, we have an associated map on sheaves $f^\# : \mathcal{O}_{\operatorname{Spec} A} \to f_*(\mathcal{O}_X)$. Taking global sections we obtain a homomorphism $A \to \mathcal{O}_X(X)$. Therefore, there is a natural map*

$$\alpha : \operatorname{Hom}(X, \operatorname{Spec} A) \to \operatorname{Hom}(A, \mathcal{O}_X(X))$$

where the first hom-set is taken in the category of schemes, and the second is taken in the category of rings. α is a bijection.

Proof. Now, the affine case follows immediately from Proposition 2.45. Therefore, it suffices to reduce to the affine case. For this, suppose that we have a ring map $\phi : A \to \mathcal{O}_X(X)$. Let $\{U_i\}$ be an affine cover of X. Then, for each i, we have a ring map $\phi_i : A \to \mathcal{O}_X(U_i)$ by composing with the restriction morphisms $\mathcal{O}_X(X) \to \mathcal{O}_X(U_i)$. Then, by the affine case, we have an induced morphism of schemes $\beta(\phi_i) : U_i \to \operatorname{Spec} A$. By gluing these maps, we get a morphism $\beta(\phi) : X \to \operatorname{Spec} A$. Hence we have a map $\beta : \operatorname{Hom}(A, \mathcal{O}_X(X)) \to \operatorname{Hom}(X, \operatorname{Spec} A)$; one may easily verify that this is the two-sided inverse of α, so α is bijective. □

Lemma 2.60 (Gluing Lemma)**.** *Let $\{X_i\}$ be a family of schemes. Suppose, for each $i \neq j$, suppose that we are given an open subset $U_{ij} \subseteq X_i$ (with the induced scheme structure). Suppose also for each $i \neq j$ we have an isomorphism of schemes $\varphi_{ij} : U_{ij} \to U_{ji}$ such that (1) for each i, j, $\varphi_{ji} = \varphi_{ij}^{-1}$, and (2) for each i, j, k, $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$, and $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_{ij} \cap U_{ik}$.*

Then there is a scheme X, together with morphisms $\psi_i : X_i \to X$ for each i, such that (1) ψ_i is an isomorphism of X_i onto an open scheme of X, (2) the $\psi_i(X_i)$ cover X, (3) $\psi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$, and (4) $\psi_i = \psi_j \circ \varphi_{ij}$ on U_{ij}. We say that X is obtained by gluing *the schemes X_i along the isomorphisms φ_{ij}.*

Corollary 2.60.1. *We define the* disjoint union *of a family of schemes* $\{X_i\}$ *by letting* U_{ij} *and* φ_{ij} *be empty for all* i, j*, and gluing together the* $\{X_i\}$ *along these empty isomorphisms. The result is denoted* $\coprod X_i$.

Next, let's investigate a way to check affineness. First, we'll need two elementary preliminary lemmas:

Lemma 2.61 (Isomorphism Criterion)**.** *Let* $f : X \to Y$ *be a morphism, and suppose* $\{U_i\}$ *is an open cover of* Y *such that the restriction* $f_i : f^{-1}(U_i) \to U_i$ *is an isomorphism for each* i*. Then* f *is an isomorphism.*

Proof. Follows immediately from basic topology and the stalk criterion for isomorphisms (Prop 2.12). □

Lemma 2.62 (Criterion for Basic Affine Open Coverings)**.** *Suppose that* A *is a ring. Then* $f_1, \ldots, f_r$ *generate* A *if and only if the basic affine opens* $D(f_i) = \operatorname{Spec}(A_{f_i})$ *cover* $\operatorname{Spec} A$.

Proof. This is a matter of definition-shuffling.

$$\varnothing = V(A) = V(\sum_{i\in I}(f_i)) = \bigcap_{i\in I} V((f_i)) \Leftrightarrow \bigcap_{i\in I} X \setminus D(f_i) = \bigcap_{i\in I} V((f_i)) = \varnothing \Leftrightarrow \bigcup_{i\in I} D(f_i) = X.$$

□

Lemma 2.63 (Criterion for Affineness)**.** *A scheme* X *is affine if and only if there is a finite set of elements* $f_1, \ldots, f_r \in \mathcal{O}_X(X)$ *such that* X_{f_i} *is affine for each* i*, and* $f_1, \ldots, f_r$ *generate the unit ideal in* $\mathcal{O}_X(X)$.

Proof. Clearly, if X is affine, then we may take $f = 1$.

Conversely, suppose there are elements $f_1, \ldots, f_r \in A = \mathcal{O}_X(X)$ such that each open subset X_{f_i} is affine and $f_1, \ldots, f_r$ generate all of $\mathcal{O}_X(X)$. Now, by Theorem 2.59, the identity map $A \to \mathcal{O}_X(X)$ induces a morphism $f : X \to \operatorname{Spec} A$. I claim that f is an isomorphism. To prove this result, recall that since the f_i generate A, the basic affine opens $D(f_i) = \operatorname{Spec}(A_{f_i})$ cover $\operatorname{Spec} A$ (this is Lemma 2.62. Now, by definition, the preimage of each open set $D(f_i)$ is X_{f_i}. By assumption, X_{f_i} is affine, isomrophic to $\operatorname{Spec} A_i$. Therefore, we have restrictions $f_i : \operatorname{Spec} A_i \to \operatorname{Spec} A_{f_i}$, where the $\operatorname{Spec} A_{f_i}$ cover $\operatorname{Spec} A$. Hence by Lemma 2.61, to show that f is an isomorphism, it suffices to show that the f_i are isomorphisms. For this, it suffices to show that the corresponding ring map (see Proposition 2.45) $\varphi : A_{f_i} \to A_i$ is an isomorphism.

Injectivity: Choose $\frac{a}{f_i^n} \in A_{f_i}$ such that $\varphi(\frac{a}{f_i^n}) = 0$. Then $\frac{a}{f_i^n}$ also vanishes in the intersection $X_{f_i} \cap X_{f_j} = \operatorname{Spec}(A_i)_{a_j}$ for each j. So for each j there is some n_j such that $f_i^{n_j} a = 0$. Then, if $m = \max\{n_1, \ldots, n_r\}$, $f_i^m a$ vanishes on each set of a cover of $\operatorname{Spec} A_{f_i}$, so $f_i^m a = 0 \in A_{f_i}$. Yet this implies that $a = 0 \in A_{f_i}$, so $\frac{a}{f_i^n} = 0 \in A_{f_i}$ as well. Therefore $\ker \varphi = 0$ and φ is injective.

Surjectivity: Take $a \in A_i$. Then, for each $j \neq i$, $\mathcal{O}_X(X_{f_i f_j}) \simeq (A_j)_{f_i}$ so $a|_{X_{f_i f_j}}$ can be written as $\frac{a_j}{f_i^{n_j}}$ for some $a_j \in A_j$. That is, we have elements $a_j \in A_j$ whose restrictions to $X_{f_i f_j}$ is $f_i^{n_j} a$. Now let $n = \max\{n_1, \ldots, n_r\}$, and replace a_j by $a_j f_i^{n-n_j}$, so that we have elements $a_j \in A_j$ whose restrictions to $X_{f_i f_j}$ is $f_i^n a$ for each j.

Now, we want to glue together these elements to a global section of X_{f_i} (that is, an element of A_{f_i}). However, they might not necessarily agree on intersections, so we have to fix that. Consider the triple intersections $X_{f_i f_j f_k} = \operatorname{Spec}(A_j)_{f_i f_k} = \operatorname{Spec}(A_k)_{f_i f_k}$; here, we have $a_j - a_k = f_i^n a - f_i^n a = 0$, and so we can find some integer m_{jk} such that $f_i^{m_{jk}}(a_j - a_k) = 0$. But then $m = \max_{1 \le i < j \le r} m_{jk}$ satisfies $f_i^m(a_j - a_k) = 0$, so the elements $f_i^m a_j$ agree on intersections and all restrict to $f_i^{n+m} a$. Hence we have a global section b whose restriction to X_{f_i} is $f_i^{n+m} a$ and so $\frac{b}{f_i^{n+m}}$ gets mapped to a by φ_i. Hence we have surjectivity. □

2.4 Nike's Trick and Types of Schemes

Following is an incredibly useful technique, which allows one to pass information from one affine cover to another. It will help us prove whenever an open cover of affine subsets of a scheme has some property P, then every affine subset has that property (examples will follow after we develop the trick).

Lemma 2.64 (Nike's Trick)**.** *Suppose that X is a scheme, and $U_1 = \operatorname{Spec}(A_1)$ and $U_2 = \operatorname{Spec}(A_2)$ are two open affine subschemes of X. Then, there exists a base $\{V_i\}$ for the topology of $U_1 \cap U_2$ such that V_i is a basic affine open in U_1 and U_2 for each i.*

Proof. Take $x \in U_1 \cap U_2$, and an open neighborhood $W \subseteq U_1 \cap U_2$ of x. Then, since basic affine opens form a base for the topology of U_1, we can pick a basic affine open $V_1 = \operatorname{Spec}(A_1)_{a_1} \subseteq W$ containing x. Then, since basic affine opens also form a base for the topology of U_2, we can pick a basic affine open $V_2 = \operatorname{Spec}(A_2)_{a_2} \subseteq V_1$ containing x. It suffices to show that V_2 is still a basic affine open of U_1.

Now, V_2 is the non-vanishing locus for the global function a_2 on U_2, because since it is contained within V_1, it is also the non-vanishing locus of $r_2|_{V_1}$ on V_1. Since $V_1 = \operatorname{Spec}(A_1)_{a_1}$, we can write $r_2|_{V_1} = \frac{a}{a_1^n}$ for some $a \in A_1$. Yet then $V_2 = \operatorname{Spec}((A_1)_{a_1})_{a/r_1^n} = \operatorname{Spec}(A_1)_{a_1 a}$, which is a basic affine open of U_1, as desired. □

Now, we will transition to looking at some definitions. These provide the aforementioned examples of properties which, if they hold for an open affine cover, hold for all affines.

Definition 2.65 ((Locally) of Finite Type)**.** A morphism $f : X \to Y$ of schemes is said to be *locally of finite type* if there exists a covering of Y by open affine subsets $V_i = \operatorname{Spec} B_i$ such that for all i, $f^{-1}(V_i)$ can be covered by open affine subsets $U_{ij} = \operatorname{Spec} A_{ij}$, where each A_{ij} is a finitely-generated B_i-algebra. f is furthermore said to be *of finite type* if we may choose each cover $\{U_{ij}\}$ of $f^{-1}(V_i)$ to be finite.

Proposition 2.66. *A morphism $f : X \to Y$ of schemes is locally of finite type if and only if for every open affine subset $V = \operatorname{Spec} B$ of Y, $f^{-1}(V)$ can be covered by open affine subsets $U_j = \operatorname{Spec} A_j$, where each A_j is a finitely generated B-algebra.*

Proof. The direction "if" is trivial. For the other direction, suppose $f : X \to Y$ is locally of finite type. Explicitly, there exists a covering of Y by open affine subsets $V_i = \operatorname{Spec} B_i$ such that for each i, $f^{-1}(V_i)$ is covered by affine opens $U_{ij} = \operatorname{Spec} A_{ij}$, where each A_{ij} is a finitely-generated B_i-algebra. Now consider any basic open affine $\operatorname{Spec}(B_i)_b \subseteq \operatorname{Spec}(B_i)$. Notice that $f^{-1}(\operatorname{Spec}(B_i)_b)$ is covered by $\operatorname{Spec}((A_{ij})_b)$, and plainly $(A_{ij})_b$ is a finitely-generated $(B_i)_b$-algebra. Hence any basic affine open of V_i satisfies the same key hypotheses as V_i, for each i. This is key to the application of Nike's trick.

Now, take $V = \operatorname{Spec} B$ to be an open affine subset. By Nike's trick, for each i, $\operatorname{Spec}(B_i) \cap \operatorname{Spec}(B)$ has an open cover $\{V_{ij}\}$ such that $V_{ij} = \operatorname{Spec} A_{ij}$ is basic affine in both $\operatorname{Spec}(B_i)$ and $\operatorname{Spec}(B)$. Notice that by our reasoning in the above paragraph, $f^{-1}(V_{ij})$ is covered by the spectra of finitely-generated A_{ij}-algebras. Yet $A_{ij} = \operatorname{Spec}(B)_{b_{ij}}$ for some $b_{ij} \in B$, so any finitely-generated A_{ij}-algebra is a finitely-generated B-algebra. Yet notice that since the V_i cover Y, the $V \cap V_i$ cover V, so the V_{ij} cover V. Hence the $f^{-1}(V_{ij})$ cover $f^{-1}(V)$, and since $f^{-1}(V_{ij})$ is covered by the spectra of finitely-generated B-alegbras, $f^{-1}(V)$ is covered by the spectra of finitely-generated B-algebras, and we are done. □

Definition 2.67 (Quasicompact Morphism)**.** A morphism $f : X \to Y$ of schemes is *quasicompact* if there is a cover of Y by open affines V_i such that $f^{-1}(V_i)$ is quasicompact for each i.

Proposition 2.68. *A morphism $f : X \to Y$ of schemes is quasicompact iff for every open affine subset $V \subseteq Y$, $f^{-1}(V)$ is quasicompact iff for every quasicompact subset $V \subseteq Y$, $f^{-1}(V)$ is quasi-compact.*

Proof. Since any affine scheme is quasicompact (see Proposition 6.28), the third condition implies the second which implies the first. Therefore, assume the first condition; we will show that the third follows. Namely, assume that there is a cover of Y by open affines V_i such that $f^{-1}(V_i)$ is quasicompact for each i.

Now fix i. Since $f^{-1}(V_i)$ is quasicompact, it is covered by finitely many affine opens $U_{ij} = \operatorname{Spec}(A_{ij})$. Then $f^{-1}((V_i)_{b_i})$ is covered by the finitely many open subschemes $\operatorname{Spec}(A_{ij})_{b_i}$, which are each quasicompact. Hence $f^{-1}((V_i)_{b_i})$ is quasicompact (see Proposition 6.27). In other words, the quasicompactness of the preimage of U_i implies the quasicompactness of any basic affine open of U_i.

Now take $V \subseteq Y$ quasicompact. Since V is covered by finitely many open affine subschemes, and the finite union of quasicompact spaces is quasicompact, it suffices to consider the affine case $V = \operatorname{Spec} B$. Now, V

is covered by $\{V \cap V_i\}$, as the V_i's cover Y. Such overlaps are open in V_i and thus covered by basic affine opens in V_i. Thus by quasi-compactness of V, there is a finite cover of V by open subschemes $U_j \subseteq \operatorname{Spec} B$ that are basic affine open in some V_i and hence have quasicompact preimage. Since $f^{-1}(V)$ is the union of the finitely many quasicompact preimages $f^{-1}(U_j)$ it is also quasicompact. □

Theorem 2.69. *A morphism $f : X \to Y$ is of finite type iff it is locally of finite type and quasicompact.*

Proof. Clearly if f is locally of finite type and quasicompact, then it is of finite type. Similarly, if f is of finite type, then it is trivially locally of finite type. Hence it suffices to show that f is quasicompact if it is of finite type. Yet this is easy: since f is of finite type, Y is covered by open affines $\{V_i\}$ whose preimages are covered by finitely many open affines $\{U_{ij}\}$. Yet by Proposition 6.27 and Proposition 6.28, this implies that each preimage $f^{-1}(V_i)$ is quasicompact, so by definition f is quasicompact. □

Corollary 2.69.1. *A morphism $f : X \to Y$ is of finite type if and only if for every open affine subset $V = \operatorname{Spec} B$ of Y, $f^{-1}(V)$ can be covered by finitely many open affine subsets $U_j = \operatorname{Spec} A_j$, where each A_j is a finitely generated B-algebra.*

Definition 2.70 (Finite Morphism)**.** A morphism $f : X \to Y$ is a *finite* morphism if there exists a covering of Y by open affine subsets $V_i = \operatorname{Spec} B_i$ such that for each i, $f^{-1}(V_i)$ is affine, equal to $\operatorname{Spec} A_i$, where A_i is a finite B_i-algebra.

The generalization of this definition to every open affine subset is a little bit more involved. In particular, we need a few algebraic lemmas:

Lemma 2.71. *Let A be a commutative ring. Suppose that $a_1, \dots, a_n \in A$ generate all of A. Then for any positive integers $m_1, \dots, m_n$, the set $a_1^{m_1}, \dots, a_n^{m_n}$ generates all of A.*

Proof. The proof is geometric: $a_1, \dots, a_n$ generates all of A if and only if $\operatorname{Spec}(A)_{a_1}, \dots, \operatorname{Spec}(A)_{a_n}$ covers $\operatorname{Spec}(A)$ (Lemma 2.62). But $\operatorname{Spec}(A)_{a_i} \subseteq \operatorname{Spec}(A)_{a_i^{m_i}}$ for each i, so $\operatorname{Spec}(A)_{a_1^{m_1}}, \dots, \operatorname{Spec}(A)_{a_n^{m_n}}$ covers $\operatorname{Spec}(A)$. Hence $a_1^{m_1}, \dots, a_n^{m_n}$ generates all of A, as desired. □

Lemma 2.72. *Suppose that A is a B-module, and there exists a finite collection $c_1, \dots, c_n$ generating B such that A_{c_i} is a finite B_{c_i}-module for each i. Then A is a finite B-module.*

Proof. Let $d_{i1}, \dots, d_{in_i}$ generate A_{c_i} as a B_{c_i}-module. By clearing denominators, notice that we may assume that $d_{i1}, \dots, d_{in_i} \in A$. I claim that the finitely many d_{ij} (ranging over all i and j) generate A. To see why, fix $a \in A$. Then for each i, we may write $a = \sum_j b_{ij} d_{ij}$ for $b_{ij} \in B_{c_i}$. Since there are finitely many b_{ij}, there exists some m_i such that $b'_{ij} = c_i^{m_i} b_{ij} \in B$ for each j. Then $c_i^{m_i} a = \sum_j b'_{ij} d_{ij}$ where $b'_{ij} \in B$ for each j.

Now, by Lemma 2.71, there exist $e_1, \dots, e_n \in B$ such that $e_1 c_1^{m_1} + \cdots + e_n c_1^{m_n} = 1$. But then,

$$\sum_i \sum_j (e_i b'_{ij}) d_{ij} = \sum_i e_i c_i^{m_i} a = e_1 c_1^{m_1} a + \cdots + e_n c_1^{m_n} a = (e_1 c_1^{m_1} + \cdots + e_n c_1^{m_n}) a = a.$$

Hence an arbitrary $a \in A$ is generated over B by the finitely many d_{ij}, so A is a finite B-module. □

Proposition 2.73. *A morphism $f : X \to Y$ is finite if and only if for every open affine subset $V = \operatorname{Spec} B$ of Y, $f^{-1}(V)$ is affine, equal to* $\operatorname{Spec} A$*, where A is a finite B-algebra.*

Proof. The direction "if" is trivial. For the other direction, suppose $f : X \to Y$ is finite. Explicitly, there exists a covering of Y by open affine subsets $V_i = \operatorname{Spec} B_i$ such that for each i, $f^{-1}(V_i)$ is affine, equal to $\operatorname{Spec} A_i$ where A_i is a finite B_i-algebra. Now, consider any basic open affine $\operatorname{Spec}(B_i)_b \subseteq \operatorname{Spec}(B_i)$. Notice that $f^{-1}(\operatorname{Spec}(B_i)_b) = \operatorname{Spec}(A_i)_b$, and plainly $(A_i)_b$ is a finite $(B_i)_b$-module. Hence any basic affine open of V_i satisfies the same key hypotheses as V_i, for each i. This is key to the application of Nike's trick.

Now, take $V = \operatorname{Spec} B$ to be an open affine subset. By Nike's trick, for each i, $\operatorname{Spec}(B_i) \cap \operatorname{Spec}(B)$ has an open cover $\{V_{ij}\}$ such that V_{ij} is basic affine in both $\operatorname{Spec}(B_i)$ and $\operatorname{Spec}(B)$. Notice that by our reasoning in the above paragraph, $f^{-1}(V_{ij})$ is affine and its corresponding ring is a finite module over the corresponding ring of V_{ij}. On the other hand, $V_{ij} = \mathrm{Spec}(B)_{c_{ij}}$ for some $c_{ij} \in B$ (since it is basic affine in Spec(B)). In summary, we have an open cover of V by $V_{ij} = \mathrm{Spec}(B)_{c_{ij}}$ such that $f^{-1}(V_{ij})$ is affine, equal to $\mathrm{Spec}(A_{ij})$ where A_{ij} is a finitely-generated $(B)_{c_{ij}}$-module. Since V is affine, it is quasicompact, so we may assume that this open cover is finite. That is, V is covered by $V_1, \ldots, V_n$ where, for each i, $V_i = \mathrm{Spec}(B)_{c_i}$ for some $c_i \in B$ and $f^{-1}(V_i) = \mathrm{Spec}(A_i)$ where A_i is a finite $(B)_{c_i}$-module.

Now define $U = f^{-1}(V) \subseteq X$ and let A equal the ring of global sections $\mathcal{O}_U(U)$. By Proposition 2.59, there is an induced map $\phi : B \to A$. Since $\mathrm{Spec}(B)_{c_1}, \ldots, \mathrm{Spec}(B)_{c_n}$ covers Spec(B), we have that $c_1, \ldots, c_n$ generate all of B. Hence there exist $b_1, \ldots, b_n \in B$ such that $b_1c_1 + \cdots + b_nc_n = 1$. But then $\phi(b_1)\phi(c_1) + \cdots + \phi(b_n)\phi(c_n) = \phi(1) = 1 \in A$, so $\phi(c_1), \ldots, \phi(c_n)$ generate all of A. Furthermore, $U_{\phi(c_i)}$ is affine as it is equal to the preimage of $\mathrm{Spec}(B)_{c_i}$, which is affine by the conclusion of the above paragraph. Hence by Lemma 2.63, U is affine and equal to Spec(A). Finally, the fact that A is a finite B-module follows immediately from Lemma 2.72 above. ☐

Definition 2.74 (Locally Noetherian and Noetherian)**.** A scheme X is called *locally Noetherian* if it can be covered by open affine subsets $\mathrm{Spec}\, A_i$, where each A_i is a Noetherian ring. X is further called *Noetherian* if it is locally Noetherian and quasicompact.

We'll use Nike's trick one last time, but again we need a few algebraic lemmas:

Lemma 2.75. *Suppose that $f_1, \ldots, f_r$ are a finite number of elements in A which generate the unit ideal, and A_{f_i} is Noetherian for each i. Then A is Noetherian.*

Proof. First, we will show that if $\mathfrak{a}$ is an ideal of A, and $\varphi_i : A \to A_{f_i}$ is the natural map,

$$\mathfrak{a} = \bigcap_i \varphi_i^{-1}(\varphi(\mathfrak{a})A_{f_i}).$$

The inclusion $\subseteq$ is obvious, so suppose that $b \in A$ is contained in the intersection. Then, for each i, we can write $\varphi_i(b) = \frac{a_i}{f_i^{n_i}}$ for some $a_i \in A$. By taking $n = \max\{n_1, \ldots, n_r\}$ and replacing a_i with $a_i f_i^{n-n_i}$, we may write $\varphi_i(b) = \frac{a_i}{f_i^n}$ for some $a_i \in A$ for all i. Then, by definition of localization, we have $f_i^{m_i}(f_i^n b - a_i) = 0$ for each i. By taking $m = \max\{m_1, \ldots, m_r\}$, we have $f_i^m(f_i^n b - a_i) = 0$ for all i. Thus $f_i^{m+n}b \in \mathfrak{a}$ for each i. But since $f_1, \ldots, f_r$ generates all of A, $f_1^{m+n}, \ldots, f_r^{m+n}$ generates all of A by Lemma 2.71. Hence there exist some $a_1, \ldots, a_r \in A$ such that $a_1 f_1^{m+n} + \cdots + a_r f_r^{m+n} = 1$. Yet then

$$b = a_1 f_1^{m+n} b + \cdots + a_r f_r^{m+n} b \in \mathfrak{a}.$$

Now that we have this result, the desired fact follows easily. Let $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \cdots$ be an ascending chain of ideals in A. Then, for each i, $\varphi_i(\mathfrak{a}_1)A_{f_i} \subseteq \varphi_i(\mathfrak{a}_2)A_{f_i} \subseteq \cdots$, becomes stationary at some N_i since A_{f_i} is Noetherian. Yet then the original chain becomes stationary at $N = \max\{N_1, \ldots, N_r\}$. Therefore every ascending chain of ideals becomes stationary in A, so A is indeed Noetherian. ☐

Proposition 2.76. *A scheme X is locally Noetherian if and only if, for every open affine subset $U = \mathrm{Spec}\, A$, A is a Noetherian ring. In particular, an affine scheme $X = \mathrm{Spec}\, A$ is a Noetherian scheme if and only if the ring A is a Noetherian ring.*

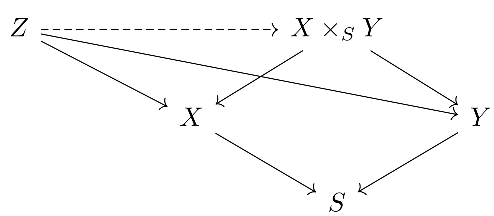
Proof. The "if" direction is trivial, and the "only if" direction is the usual use of Nike's Lemma. That is, assume that X is locally Noetherian; i.e., that there is an open affine cover $U_i = \mathrm{Spec}\, A_i$ of X such that A_i is Noetherian for each i. Notice that since the localization of any Noetherian ring is Noetherian, any basic affine open of U_i is the spectrum of a Noetherian ring. Now let $U = \mathrm{Spec}\, A$ be an open affine subset of X. For each i, $U_i \cap U$ is covered by $\{V_{ij}\}$ where V_{ij} is a basic affine open of both U_i and U. Now, since V_{ij} is a basic affine open of U_i, it has Noetherian coordinate ring. Furthermore, since the U_i cover X, the $U_i \cap U$ cover U, so the V_{ij} cover U. Since U is affine, it is quasicompact, so a finite collection $V_1, \ldots, V_r$ cover U. Now, $V_i = \mathrm{Spec}(A_{f_i})$ for some $f_i \in A$ since it is a basic affine open in U. Since the $D(f_i) = \mathrm{Spec}(A_{f_i})$ cover U, they must generate the unit ideal (Lemma 2.62). Furthermore, A_{f_i} is Noetherian for each i by hypothesis. Therefore, by Lemma 2.75, A is Noetherian, and we are done.

Now, if $X = \operatorname{Spec} A$ is a Noetherian scheme, obviously the ring $A = \mathcal{O}_X(X)$ must be Noetherian. On the other hand, if $A = \mathcal{O}_X(X)$ is Noetherian, then $\{X\}$ suffices as an open cover of X by spectra of Noetherian rings, so X is locally Noetherian. Furthermore, as an affine scheme, X is quasicompact (see Proposition 6.28), so it is by definition Noetherian. □

Definition 2.77 (Scheme over S)**.** Let S be a fixed scheme. A *scheme over* S is a scheme X, together with a morphism $X \to S$. If X and Y are schemes over S, a *morphism of* X *to* Y *as schemes over* S (also called an S*-morphism*) is a morphism $f : X \to Y$ compatible with the structure morphisms to S.

Definition 2.78 (Fibred Product)**.** Let S be a scheme, and let X and Y be schemes over S. Then the *fibred product* of X and Y over S, denoted $X \times_S Y$, is a scheme, together with morphisms $p_1 : X \times_S Y \to X$ and $p_2 : X \times_S Y \to Y$, which make a commutative diagram with the structure morphisms $X \to S$ and $Y \to S$, satisfying the following universal property:

> Given any scheme Z over S, and given morphisms $f : Z \to X$ and $g : Z \to Y$ which make a commutative diagram with the structure morphisms $X \to S$ and $Y \to S$, there exists a unique morphism $\theta : Z \to X \times_S Y$ such that $f = p_1 \circ \theta$ and $g = p_2 \circ \theta$; that is, we have the following commutative diagram:



The morphisms p_1 and p_2 are called the *projection morphisms* of the fibred product onto its factors.

Theorem 2.79 (The Fibred Product Exists)**.** *For any two schemes X and Y over a scheme S, the fibred product $X \times_S Y$ exists, and is unique up to unique isomorphism.*

Proof. Pg. 87-88 of Hartshorne. □

Definition 2.80 (The Fibre of a Morphism)**.** Let $f : X \to Y$ be a morphism of schemes, and let $y \in Y$ be a point. Let $k(y)$ be the residue field of y, and let $\operatorname{Spec} k(y) \to Y$ be the natural morphism. Then we define the *fibre* of the morphism f over the point y to be the scheme

$$X_y = X \times_Y \operatorname{Spec} k(y).$$

Proposition 2.81. *If $f : X \to Y$ is a morphism, and $y \in Y$ is a point, then $\operatorname{sp}(X_y)$ is homeomorphic to $f^{-1}(y)$ with the induced topology.*

Proof. First notice that by replacing Y with an open affine subset of Y containing y, we may assume that Y is affine, say equal to $\operatorname{Spec} A$. Then, we will reduce to the case where X is affine. Take an open affine cover $\{U_i\}$ of X, with U_i. Then, we apply the fact that fibred products are constructed from the glueing of affine products (see Steps 3-5 of Theorem 3.3 of Hartshrone) to see that

$$X_y = X \times_Y \kappa(y) = \left(\bigcup_i U_i \times_Y \kappa(y) \right) = \bigcup_i (U_i \times_Y \kappa(y)) = \bigcup_i f^{-1}|_{U_i}(y) = f^{-1}(y)$$

as *topological spaces*, where the fourth equality is exactly the affine case. Therefore it suffices to show the affine case; that is, we may assume that both X and Y are affine with $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$.

In this case, y is a prime ideal $\mathfrak{p} \triangleleft_{\mathrm{pr}} A$, and from Step 1 of Theorem 3.3 of Hartshorne we have that $X_y = \operatorname{Spec}(B \otimes_A \kappa(\mathfrak{p}))$. Next, define $S = A \setminus \mathfrak{p}$, and notice that

$$B \otimes_A \kappa(\mathfrak{p}) = B \otimes_A A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} = B \otimes_A A_{\mathfrak{p}} \otimes_A A/\mathfrak{p} = S^{-1}B \otimes A/\mathfrak{p} = S^{-1}B/\mathfrak{p}S^{-1}B.$$

Yet $\operatorname{Spec}(S^{-1}B) = \{\mathfrak{q} \in \operatorname{Spec} B \mid f^{-1}(\mathfrak{q}) \subseteq \mathfrak{p}\}$, so by the correspondence of prime ideals in quotients,

$$\operatorname{Spec}(S^{-1}B/\mathfrak{p}S^{-1}B) = \{\mathfrak{q} \in \operatorname{Spec} B \mid f^{-1}(\mathfrak{q}) \subseteq \mathfrak{p}, f(\mathfrak{p}) \subseteq \mathfrak{q}\}.$$

Yet notice that $f(\mathfrak{p}) \subseteq \mathfrak{q}$ implies that $\mathfrak{p} \subseteq f^{-1}(f(\mathfrak{p})) \subseteq f^{-1}(\mathfrak{q})$, so in fact

$$\operatorname{Spec}(S^{-1}B/\mathfrak{p}S^{-1}B) = \{\mathfrak{q} \in \operatorname{Spec} B \mid f^{-1}(\mathfrak{q}) \subseteq \mathfrak{p} \subseteq f^{-1}(\mathfrak{q})\} = \{\mathfrak{p} \in \operatorname{Spec} B \mid f^{-1}(\mathfrak{q}) = \mathfrak{p}\} = f^{-1}(\mathfrak{p}).$$

Hence $X_y = f^{-1}(y)$ in the induced topology, completing the affine case, so we are done. □

Definition 2.82 (Quasi-Finite Morphism). A morphism $f : X \to Y$ is called *quasi-finite* if for every point $y \in Y$, $f^{-1}(y)$ is a finite set.

Definition 2.83 (Dominant Morphism). A morphism $f : X \to Y$ of schemes is called *dominant* if the image of f is a dense subset of Y.

Earlier, we discussed open subschemes. As it turns out, there is a notion of a "closed subscheme", though it is more complex.

Definition 2.84 (Closed Immersion and Subscheme). A *closed immersion* is a morphism $f : Y \to X$ of schemes such that f induces a homeomorphism of $\operatorname{sp}(Y)$ onto a closed subset of $\operatorname{sp}(X)$, and furthermore the induced map $f^{\#} : \mathcal{O}_X \to f_*\mathcal{O}_Y$ of sheaves on X is surjective. A *closed subscheme* of a scheme X is an equivalence class of closed immersions, where we say $f : Y \to X$ and $f' : Y' \to X$ are equivalent if there is an isomorphism $i : Y' \to Y$ compatible with the immersions f and f' (that is, such that $f' = f \circ i$).

Following is a natural example of a closed subscheme:

Theorem 2.85 (Closed Subscheme of Affine Scheme). *If Y is a closed subscheme of an affine scheme $X = \operatorname{Spec} A$, then Y is also affine; furthermore, Y is the closed subscheme determined by a suitable ideal $\mathfrak{a} \subseteq A$ as the image of the closed immersion* $\operatorname{Spec} A/\mathfrak{a} \to \operatorname{Spec} A$.

Proof. Identify Y with its homeomorphic closed image in X. Firstly, recall that any open subset of Y has the form $U \cap Y$ for some open subset U of X. Then, since Y is covered by open affine sets, there is some open affine cover $\{U_i \cap Y\}_{i\in I}$ of Y where $U_i \subseteq X$ is open for each i. Now, since the basic open affine sets form a base of the topology on $\operatorname{Spec} A$, each U_i is the union of basic open affine sets. Therefore, we can replace each U_i in the open affine cover $\{U_i \cap Y\}$ by some $D(f_j)$; in other words, we have a new cover $\{D(f_j) \cap Y\}_{j\in J}$ of Y. I claim that $D(f_j) \cap Y$ is affine for each j. To see why, fix j and notice that $D(f_j) = (U_i)_{f_j}$ (where U_i is a set which was replaced by $D(f_j)$ among others) , so $D(f_j) \cap Y = (U_i)_{f_j} \cap Y = (U_i \cap Y)_{f_j}$, which is affine because $U_j \cap Y$ was. Hence, we have an open affine cover of Y of the form $\{D(f_j) \cap Y\}_{j\in J}$, where the f_j are in A.

Now, since Y is closed, $X \setminus Y$ is open, so it is the union of some basic open affine sets. In other words, we may enlarge the cover of Y by adding more f_j until the collection $\{D(f_j)\}$ covers all of X. Since the f_j we add are such that $D(f_j) \cap Y = \varnothing$, the open cover $\{D(f_j) \cap Y\}_{j\in J}$ of Y is still affine. Now, since X is quasicompact, we may assume that there are only finitely many $j \in J$. In summary, we have elements $f_1, \dots, f_n$ such that $\{D(f_1) \cap Y, \dots, D(f_n) \cap Y\}$ is an open affine cover of Y, and $D(f_1), \dots, D(f_n)$ is an open affine cover of X. From here, we can quickly conclude the result.

Firstly, by Lemma 2.62, $f_1, \dots, f_n$ generate all of A. Yet, by the definition of a closed immersion $f : Y \to X$, the map $f^{\#} : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective; in particular, we have a surjective map π from A (which is the ring of global sections of X) to the ring of global sections of Y, which we call B. Clearly, $\pi(f_1), \dots, \pi(f_n)$ generate all of B. Hence by Lemma 2.63, Y is affine and equals $\operatorname{Spec} B$. Furthermore, by assigning $\mathfrak{a} = \ker \pi$, we may conclude that $B \simeq A/\mathfrak{a}$ by the First Isomorphism Theorem, as desired. □

2.5 Seperated and Proper Morphisms

Definition 2.86 (Diagonal Morphism and Separated Morphisms). Let $f : X \to Y$ be a morphism of schemes. The *diagonal morphism* is the unique morphism $\Delta : X \to X \times_Y X$ whose composition with both projection maps $p_1, p_2 : X \times_Y X \to X$ is the identity map id_X. We say that f is *separated* if the diagonal morphism Δ is a closed immersion. In that case, we also say X is *separated* over Y.

Definition 2.87 (Separated Schemes)**.** By Theorem 2.59, since $\mathbb{Z}$ is an initial object in the category of rings, $\operatorname{Spec}\mathbb{Z}$ is a terminal object in the category of schemes. Hence any scheme X comes with a unique map $X \to \operatorname{Spec}\mathbb{Z}$. X is said to be *separated* if this map is separated.

Proposition 2.88. *If $f : X \to Y$ is any morphism of affine schemes, then f is separated.*

Proof. Let $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$. Then A is a B-algebra, and $X \times_Y X$ is also affine, given by $\operatorname{Spec} A \otimes_B A$ (this is easily proven using the universal properties of fibre products and tensor products and Proposition 2.59). The diagonal morphism Δ comes from the *diagonal homomorphism* $A \otimes_B A \to A$ defined by $a \otimes a' \to aa'$. This is a surjective homomorphism of rings, hence Δ is a closed immersion. ☐

Corollary 2.88.1. *An arbitrary morphism $f : X \to Y$ is separated if and only if the image of the diagonal morphism is a closed subset of $X \times_Y X$.*

Proof. One implication is obvious, so it suffices to prove that if $\Delta(X)$ is a closed subset, then $\Delta : X \to X \times_Y X$ is a closed immersion.

First, we will check that $\Delta : X \to \Delta(X)$ is a homeomorphism. Let $p_1 : X \times_Y X \to X$ be the first projection. Since $p_1 \circ \Delta = \mathrm{id}_X$, Δ is injective; it is clearly surjective onto $\Delta(X)$, so it is a bijection. Furthermore, Δ is obviously continuous, and its inverse is p_1, which is also continuous.

Next, we will check that $\Delta^\# : \mathcal{O}_{X\times_Y X} \to \Delta_*(\mathcal{O}_X)$ is surjective. Using Proposition 2.25, we see that this is a local question; that is, $\Delta^\#$ is surjective if and only if $\Delta^\#_x$ is surjective for each x. Yet each such stalk map is also the stalk map of Δ restricted to an open affine subset (which is surjective by the above proposition). Hence every stalk map is indeed surjective, and we are done. ☐

Theorem 2.89 (Valuative Criterion of Separatedness)**.** *Let $f : X \to Y$ be a morphism of schemes, and assume that X is Noetherian. Then f is separated if and only if the following condition holds:*

For any field K, and for any valuation ring R of K, let $T = \operatorname{Spec} R$, $U = \operatorname{Spec} K$, and $i : U \to T$ be the morphism induced by the inclusion $R \hookrightarrow K$. Given morphisms $T \to Y$ and $U \to X$ which makes a commutative diagram, there is at most one morphism $T \to X$ making the whole diagram commute.

$$\begin{array}{ccc} U & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ T & \longrightarrow & Y \end{array}$$

Proof. Pg. 97-99 of Hartshorne. ☐

Corollary 2.89.1. *Assume that all schemes are Noetherian in the following statements.*

(a) Open and closed immersions are separated.

(b) A composition of two separated morphisms is separated.

(c) Separated morpshisms are stable under base extension.

(d) If $f : X \to Y$ and $f' : X' \to Y'$ are separated morphisms of schemes over a base scheme S, then the morphism $f \times f' : X \times_S X' \to Y \times_S Y'$ is also separated.

(e) If $f : X \to Y$ and $g : Y \to Z$ are two morphisms and if $g \circ f$ is separated, then f is separated.

(f) A morphism $f : X \to Y$ is separated if and only if Y can be covered by open subsets V_i such that $f^{-1}(V_i) \to V_i$ is separated for each i.

Proof. Immediate from applying the above criterion. ☐

Definition 2.90 (Base Extension)**.** Let $f : X \to Y$ be a morphism of schemes. Then, given a morphism $Y' \to Y$, and letting $X' = X \times_Y Y'$, we are given a morphism $f : X' \to Y' \times_Y Y = Y'$, called the *base change* of f by the morphism $Y' \to Y$.

Definition 2.91 (Proper)**.** A morphism $f : X \to Y$ is *universally closed* if it is closed, and for any morphism $Y' \to Y$, the corresponding morphism $f' : X' \to Y'$ obtained by base extension is also closed. A morphism $f : X \to Y$ is *proper* if it is separated, of finite type, and universally closed.

Theorem 2.92 (Valuative Criterion of Properness)**.** *Let $f : X \to Y$ be a morphism of schemes, and assume that X is Noetherian. Then f is separated if and only if the following condition holds:*

For any field K, and for any valuation ring R of K, let $T = \operatorname{Spec} R$, $U = \operatorname{Spec} K$, and $i : U \to T$ be the morphism induced by the inclusion $R \hookrightarrow K$. Given morphisms $T \to Y$ and $U \to X$ which makes a commutative diagram, there exists a unique *morphism $T \to X$ making the whole diagram commute.*

$$\begin{array}{ccc} U & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ T & \longrightarrow & Y \end{array}$$

Proof. Pg. 101-102 in Hartshorne. □

Corollary 2.92.1. *Assume that all schemes are Noetherian in the following statements.*

(a) A closed immersion is proper.

(b) A composition of proper morphisms is proper.

(c) Proper morpshisms are stable under base extension.

(d) If $f : X \to Y$ and $f' : X' \to Y'$ are proper morphisms of schemes over a base scheme S, then the morphism $f \times f' : X \times_S X' \to Y \times_S Y'$ is also proper.

(e) If $f : X \to Y$ and $g : Y \to Z$ are two morphisms, $g \circ f$ is proper, and g is separated, then f is proper.

(f) A morphism $f : X \to Y$ is proper if and only if Y can be covered by open subsets V_i such that $f^{-1}(V_i) \to V_i$ is proper for each i.

Proof. Immediate from applying the above criterion. □

2.6 Module Sheaves and (Quasi)coherence

Definition 2.93 ($\mathcal{O}_X$-Modules)**.** Let $(X, \mathcal{O}_X)$ be a ringed space. A $\mathcal{O}_X$-module is a sheaf $\mathscr{F}$ on X such that (1) for each open set $U \subseteq X$, $\mathscr{F}(U)$ is an $\mathcal{O}_X$-module, and (2) for any inclusion of open sets $U \subseteq V$, the restriction homomorphism $\mathscr{F}(U) \to \mathscr{F}(V)$ are compatible with the restriction maps $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$; that is, $(fs)|_V = f|_V s|_V$.

Definition 2.94 (Morphism of $\mathcal{O}_X$-Modules)**.** A *morphism* $\mathscr{F} \to \mathscr{G}$ of sheaves of $\mathcal{O}_X$-modules is a morphism of sheaves such that for each open set $U \subseteq X$, the map $\mathscr{F}(U) \to \mathscr{G}(U)$ is a homomorphism of $\mathcal{O}_X(U)$-modules.

Definition 2.95 (Examples of $\mathcal{O}_X$-Modules)**.**

(1) Suppose that $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of $\mathcal{O}_X$-modules. Then $\ker \varphi$, $\operatorname{im} \varphi$, and $\operatorname{coker} \varphi$ are all $\mathcal{O}_X$-modules.

(2) Suppose that $\mathscr{F}'$ is a $\mathcal{O}_X$-submodule of the $\mathcal{O}_X$-module $\mathscr{F}$ (that is, $\mathscr{F}'$ is a $\mathcal{O}_X$-module and a subsheaf of $\mathscr{F}$). Then $\mathscr{F}/\mathscr{F}'$ is a $\mathcal{O}_X$-module.

(3) Any direct sum, direct product, direct limit, or inverse limit of $\mathcal{O}_X$-modules is a $\mathcal{O}_X$-module.

(4) If $\mathscr{F}$ and $\mathscr{G}$ are two $\mathcal{O}_X$-modules, the sheaf $\mathscr{H}om$ $U \mapsto \operatorname{Hom}_{\mathcal{O}_X|_U}(\mathscr{F}|_U, \mathscr{G}|_U)$ is a $\mathcal{O}_X$-module.

(5) The *tensor product* of two $\mathcal{O}_X$-modules $\mathscr{F}$ and $\mathscr{G}$, denoted $\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G}$, is the sheafification of the presheaf $U \mapsto \mathscr{F}(U) \otimes_{\mathcal{O}_X(U)} \mathscr{G}(U)$.

Definition 2.96 (Free and Locally Free $\mathcal{O}_X$-Modules)**.** An $\mathcal{O}_X$-module $\mathscr{F}$ is *free* if it is isomorphic to a direct sum of copies of $\mathcal{O}_X$. In this case, the rank of $\mathscr{F}$ is the number of copies of $\mathcal{O}_X$ needed. On the other hand, $\mathscr{F}$ is *locally free* if there exists an open cover $\{U_i\}$ for which $\mathscr{F}|_{U_i}$ is a free $\mathcal{O}_X|_{U_i}$-module. In that case, the rank of $\mathscr{F}$ may depend on the open set, but when X is connected it is the same everywhere. A locally free sheaf of rank 1 everywhere is called an *invertible sheaf.*

Definition 2.97 (Sheaf of Ideals)**.** A *sheaf of ideals* on X is a sheaf of modules $\mathscr{J}$ which is a subsheaf of $\mathcal{O}_X$. In other words, for every open set U, $\mathscr{J}(U)$ is an ideal of $\mathcal{O}_X(U)$.

Definition 2.98 (Direct and Inverse Images of $\mathcal{O}_X$-Modules)**.** Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. If $\mathscr{F}$ is an $\mathcal{O}_X$-module, then $f_*\mathscr{F}$ is an $f_*\mathcal{O}_X$-module. Since we have a morphism $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$, this makes $f_*\mathscr{F}$ naturally a $\mathcal{O}_Y$-module, called the *direct image* of $\mathscr{F}$ by f.

On the other hand, let $\mathscr{G}$ be a sheaf of $\mathcal{O}_Y$-modules. Then $f^{-1}\mathscr{G}$ is an $f^{-1}\mathcal{O}_Y$-module. Now, we have a morphism $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. Then, define the *inverse image* $f^*\mathscr{G}$ to be the tensor product $f^{-1}\mathscr{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$, which is naturally a $\mathcal{O}_X$-module.

We will not prove this fact, but f_* and f^* are adjoint functors between the category of $\mathcal{O}_X$-modules and the category of $\mathcal{O}_Y$-modules.

Definition 2.99 (Sheaf Associated To a Module)**.** Let A be a ring and M an A-module. Then $\widetilde{M}$, the *sheaf associated to* M, is defined as follows:

> For any open $U \subseteq \operatorname{Spec} A$, define the group $\widetilde{M}(U)$ to be the set of functions $s : U \to \coprod_{\mathfrak{p} \in U} M_\mathfrak{p}$ such that for each $\mathfrak{p} \in U$, $s(\mathfrak{p}) \in M_\mathfrak{p}$, and such that s is locally a fraction m/f with $m \in M$ and $f \in A$ (see Definition 2.43 for the precise meaning of this statement). We also made $\widetilde{M}$ into a sheaf by using the obvious restriction maps.

Then $\widetilde{M}$ is clearly an $\mathcal{O}_{\operatorname{Spec} A}$-module.

Theorem 2.100 (Properties of the Sheaf Associated To a Module)**.** *Let A be a ring and M be an A-module.*

(1) For each $\mathfrak{p} \in \operatorname{Spec} A$, the stalk $(\widetilde{M})_\mathfrak{p}$ of the sheaf $\widetilde{M}$ at $\mathfrak{p}$ is isomorphic to $M_\mathfrak{p}$.

(2) For any $f \in A$, the A_f-module $\widetilde{M}(D(f))$ is isomorphic to M_f. In particular, $\widetilde{M}(X) = M$.

(3) The map $M \to \widetilde{M}$ is an exact, fully faithful functor from the category of A-modules to the category of $\mathcal{O}_X$-modules.

(4) If M and N are two A-modules, then $(\widetilde{M \otimes_A N}) = \widetilde{M} \otimes \widetilde{N}$.

(5) If $\{M_i\}$ is a family of A-modules, then $(\widetilde{\bigoplus_i M_i}) \simeq \bigoplus_i \widetilde{M_i}$.

(6) Let $A \to B$ be a ring homomorphism and $f : \operatorname{Spec} B \to \operatorname{Spec} A$ be the corresponding morphism of spectra. Then, any B-module N is naturally an A-module, and $f_*(\widetilde{N}) \simeq \widetilde{N}$ as $\mathcal{O}_{\operatorname{Spec} A}$-modules.

(7) Let $A \to B$ be a ring homomorphism and $f : \operatorname{Spec} B \to \operatorname{Spec} A$ be the corresponding morphism of spectra. Then, for any A-module M, $f^*(\widetilde{M}) \simeq (\widetilde{M \otimes_A B})$ as $\mathcal{O}_{\operatorname{Spec} B}$-modules.

Proof. The proofs of (1) and (2) are analogous to the proofs in the case of rings. To prove (3), notice that the functor is exact because localization and exact and exactness of sheaves can be measured at the stalks, which are computed by localization. Furthermore, it is fully faithful (i.e. $\operatorname{Hom}_A(M, N) \simeq \operatorname{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$ because the functor gives a natural map $\operatorname{Hom}_A(M, N) \to \operatorname{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$ and evaluation at global sections gives a natural inverse map in the opposite direction.

(4) and (5) follow immediately because direct sum and tensor product commute with localization, so one may quickly compute that the natural maps are isomorphisms. Finally, (6) and (7) follow from the definitions. □

Definition 2.101 (Quasicoherent and Coherent $\mathcal{O}_X$-Modules)**.** Let $(X, \mathcal{O}_X)$ be a scheme. A sheaf of $\mathcal{O}_X$-modules $\mathscr{F}$ is *quasicoherent* if X can be covered by open affine subsets $U_i = \operatorname{Spec} A_i$ such that for each i there is an A_i-module M_i with $\mathscr{F}|_{U_i} \simeq \widetilde{M_i}$. We say that $\mathscr{F}$ is *coherent* if furthermore each M_i can be taken to be a finitely generated A_i-module.

Lemma 2.102 (Global and Restricted Sections of $\mathcal{O}_X$-Modules)**.** *Let A be a ring, take $f \in A$, and let $\mathscr{F}$ be a quasicoherent sheaf on X. Then,*

(a) If $s \in \mathscr{F}(X)$ is a global section of $\mathscr{F}$ whose restriction to $D(f)$ is 0, then for some $n > 0$, $f^n(s) = 0$.

(b) Given a section $t \in \mathscr{F}(D(f))$ of $\mathscr{F}$ over the open set $D(f)$, then for some $n > 0$, $f^n(t)$ extends to a global section of $\mathscr{F}$ over X.

Proof. First, since $\mathscr{F}$ is quasicoherent, by definition X can be covered by open affine subsets $\{U_i\}$ (with $U_i = \operatorname{Spec} A_i$) such that for each i there is an A_i-module M_i with $\mathscr{F}|_{U_i} = \widetilde{M_i}$. Now, recall that the basic affine opens $D(f)$ form a base for the topology of X, so for each i, U_i is the union of $D(f_{ij})$ for various $f_{ij} \in A$. Now, the inclusion map $D(f_{ij}) \hookrightarrow U_i$ corresponds to a ring homomorphism $U_i \to A_{f_{ij}}$, making $A_{f_{ij}}$ into a U_i-module. Therefore, consider the $A_{f_{ij}}$-module $M_i \otimes_{A_i} A_{f_{ij}}$; notice that $\mathscr{F}|_{D_g} \simeq M_i \widetilde{\otimes_{A_i}} A_{f_{ij}}$.

Thus, by renumbering, if $\mathscr{F}$ is quasicoherent on X then X can be covered by basic affine opens $D(f_j)$ such that, for each j, $\mathscr{F}|_{D(f_j)} \simeq \widetilde{M_j}$ for some module M_j over the ring A_{f_j}. As an affine scheme, X is quasicompact, so we may assume there are only finitely many j.

(a): Now suppose $s \in \mathscr{F}(X)$ satisfies $s|_{D(f)} = 0$. Then, for each j, $s|_{D(f_j)}$ is an element of M_j. Furthermore, $D(f) \cap D(f_j) = D(ff_j)$, so $\mathscr{F}|_{D(ff_j)} = \widetilde{(M_i)_f}$ by Theorem 2.100. Now, the image of s in $(M_i)_f$ is zero, so by the definition of localization $f^{n_j}(s|_{D(f_j)}) = 0$ for some n_j. Now, since there are finitely many j, we may choose n larger than all the n_j, so that $f^n s$ restricts to 0 on each $D(f_j)$ and therefore $f^n s = 0$.

(b): Take $t \in \mathscr{F}(D(f))$. Then, as before, for each j we may restrict t to $\mathscr{F}(D(ff_j)) = (M_j)_f$. By the definition of localization, for some $n > 0$ there is an element $t_j \in M_j = \mathscr{F}(D(f_j))$ which restricts to $f^{n_j} t$ on $D(ff_j)$. Again, we choose n to be larger than all the n_j, so that for each j we have an element $t_j \in M_j = \mathscr{F}(D(f_j))$ such that t_j restricts to $f^n t$ on $D(ff_j)$. Now, we seek to glue the t_j together to a global section of $\mathscr{F}$. However, they may not necessarily agree. Therefore, consider the intersection $D(f_j) \cap D(f_k) = D(f_j f_k)$. Now, in $D(ff_j f_k)$, t_j and t_k are equal to $f^n t$; in particular, there their difference is 0, so there is an integer m_{jk} such that $f^{m_{jk}}(t_j - t_k) = 0$ in $D(f_j f_k)$ by part (a). Choose m to be larger than all the m_{jk}, so that $f^m(t_j - t_k) = 0$ in $D(f_j f_k)$ for all j, k. Then the $f^m t_j$ are compatible for all j, so they glue together a global section s of $\mathscr{F}$ whose restriction to $D(f)$ is $f^{n+m} t$. ☐

Theorem 2.103 (Stronger Notion of (Quasi)coherence)**.** *Let X be a scheme. Then a $\mathcal{O}_X$-module $\mathscr{F}$ is quasicoherent if and only if for* every *open affine subset $U = \operatorname{Spec} A$ of X, there is an A-module M such that $\mathscr{F}|_U \simeq \widetilde{M}$. If X is also Noetherian, if $\mathscr{F}$ is coherent if and only if for* every *open affine subset $U = \operatorname{Spec} A$ of X, there is a finitely-generated A-module M such that $\mathscr{F}|_U \simeq \widetilde{M}$.*

Proof. Let $\mathscr{F}$ be quasicoherent on X and let $U = \operatorname{Spec} A$ be an open affine. As in the proof of Lemma 2.102, there is a base for the topology consisting of open affines for which the restriction of $\mathscr{F}$ is the sheaf associated to a module. In particular, $\mathscr{F}|_U$ is quasicoherent, so we may assume that X is affine, say equal to $\operatorname{Spec} A$.

Now, let $M = \mathscr{F}(X)$; it suffices to show that $\mathscr{F} = \widetilde{M}$. For this, consider the natural map $\alpha : \widetilde{M} \to \mathscr{F}$; we will show it is an isomorphism. Now, as in the proof of the preceding lemma, X can be covered by open sets $D(f_j)$ with $\mathscr{F}|_{D(f_j)} \simeq \widetilde{M_j}$ for some A_{f_j}-module M_j. Now, the lemma, applied to the open set $D(f_j)$, tells us that $\mathscr{F}(D(f_j)) = M_{f_j}$, so $M_i = M_{f_j}$. It follows that α restricted to $D(f_j)$ is an isomorphism, and since the $D(f_j)$ cover X, α is an isomorphism. Now, suppose X is Noetherian and $\mathscr{F}$ is coherent. Then the remainder of the result follows from Lemma 2.72. ☐

Corollary 2.103.1. *Let A be a ring with spectrum $X = \operatorname{Spec} A$. Then the functor $M \mapsto \widetilde{M}$ gives an equivalence of categories between the category of A-modules and the category of quasicoherent $\mathcal{O}_X$-modules* *(the inverse is given by $\widetilde{M} \mapsto \widetilde{M}(X)$. Furthermore, if A is a Noetherian ring, $M \mapsto \widetilde{M}$ gives an equivalence of categories between the category of finitely-generated A-modules and the category of coherent $\mathcal{O}_X$-modules (with the same inverse).*

Proposition 2.104 (Exactness of Evaluation)**.** *Let X be an affine scheme, $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ be an exact sequence of $\mathcal{O}_X$-modules, and assume that $\mathscr{F}'$ is quasi-coherent. Then the sequence $0 \to \mathscr{F}'(X) \to \mathscr{F}(X) \to \mathscr{F}''(X) \to 0$ is exact.*

Proof. By Theorem 2.30, it suffices to show that $\mathscr{F}(X) \to \mathscr{F}''(X)$ is surjective. Let $s \in \mathscr{F}''(X)$; since $\mathscr{F} \to \mathscr{F}''$ is surjective, by Proposition 2.26 and the quasicompactness of X, there are finitely many $f_1, \ldots, f_n$ such that $s|_{D(f_i)}$ lifts to a section $t_i \in \mathscr{F}(D(f_i))$ for each i and $D(f_1), \ldots, D(f_n)$ covers X. We may force the t_i to be compatible without changing where they map, and then glue them together into a global section $t \in \mathscr{F}(X)$ which maps to s, as desired. □

Proposition 2.105 (Examples of Quasicoherent Sheaves)**.**

(1) The kernel, cokernel, and image of any morphism of quasicoherent sheaves are quasicoherent. If X is Noetherian, the same is true for coherent sheaves.

(2) Any extension of quasicoherent sheaves is quasicoherent. If X is Noetherian, the same is true for coherent sheaves.

(3) If $f : X \to Y$ is a morphism of schemes, and $\mathscr{G}$ is a quasicoherent $\mathcal{O}_Y$-module, then $f^*\mathscr{G}$ is a quasicoherent $\mathcal{O}_X$-module. If X and Y are Noetherian, and $\mathscr{G}$ is coherent, then $f^*\mathscr{G}$ is also coherent.

(4) Let $f : X \to Y$ be a morphism of schemes, and assume either that X is Noetherian or f is quasi-compact and seperated. Then, if $\mathscr{F}$ is a quasicoherent $\mathcal{O}_X$-module, $f_*\mathscr{F}$ is a quasicoherent $\mathcal{O}_Y$-module.

Proof. As previously discussed (in say Lemma 2.102 and Theorem 2.103), the question is local, so we may assume that $X = \operatorname{Spec} A$ is affine for (1) and (2). Also, throughout, we will skip the proofs of the additional "coherent if coherent over Noetherian" statements, as they are generally very simple.

(1) immediately follows from Theorem 2.100(3) and Theorem 2.103.

For (2), notice that given an exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ of $\mathcal{O}_X$-modules with $\mathscr{F}'$ and $\mathscr{F}''$ quasicoherent, we get an exact sequence $0 \to M' \to M \to M'' \to 0$ by taking global sections. Then, by Theorem 2.100(c), we get an exact sequence $0 \to \widetilde{M'} \to \widetilde{M} \to \widetilde{M''} \to 0$. Now, adding the natural maps, we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{M'} & \longrightarrow & \widetilde{M} & \longrightarrow & \widetilde{M''} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{F}' & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{F}'' & \longrightarrow & 0 \end{array}$$

where the two outside arrows are isomorphisms, so by the 5-lemma, the middle one is also, showing that $\mathscr{F}$ is quasi-coherent. Hence any extension of quasicoherent sheaves is quasicoherent.

(3)-(4): Pg. 116 of Hartshorne. □

Definition 2.106 (Ideal Sheaf of Closed Subscheme)**.** Let Y be a closed subscheme of a scheme X, and let $i : Y \to X$ be the inclusion morphism. Then we define the *ideal sheaf* of Y, denoted $\mathscr{J}_Y$, to be the kernel of the morphism $i^{\#} : \mathcal{O}_X \to i_*\mathcal{O}_Y$.

Proposition 2.107 (Properties of Ideal Sheaves)**.** *Let X be a scheme. For any closed subscheme Y of X, the corresponding ideal sheaf $\mathscr{J}_Y$ is a quasicoherent sheaf of ideals of X. If X is Noetherian, then $\mathscr{J}_Y$ is also coherent. Conversely, any quasicoherent sheaf of ideals on X is the ideal sheaf of a uniquely determined closd subscheme of X.*

Proof. If Y is a closed subscheme of X, then the inclusion morphism $i : Y \to X$ is quasicompact (as it is injective) and separated by Corollary 2.89.1. Hence, by Proposition 2.105, $\mathscr{J}_Y$ is the kernel of a morphism of quasicoherent sheaves and hence is quasicoherent. If X is Noetherian, then for any open affine subset $U \subseteq X$, the coordinate ring of U is Noetherian, so the ideal $I = \mathscr{J}_Y|_U(U)$ is finitely generated. Hence, in this case, $\mathscr{J}_Y$ is coherent.

Conversely, given a scheme X and a quasicoherent sheaf of ideals $\mathscr{J}$, let Y be the support of the quotient sheaf $\mathcal{O}_X/\mathscr{J}$. Then Y is a subspace of X; I claim that $(Y, \mathcal{O}_X/\mathscr{J})$ is the unique closed subscheme of X with ideal sheaf $\mathscr{J}$. The uniqueness is clear, so we have only to check that $(Y, \mathcal{O}_X/\mathscr{J})$ is a closed subscheme. This is a local question, so assume that $X = \operatorname{Spec} A$ is affine. Since $\mathscr{J}$ is quasicoherent, this implies that $\mathscr{J} = \widetilde{\mathfrak{a}}$ for some ideal $\mathfrak{a} \lhd A$. Then $(Y, \mathcal{O}_X/\mathscr{J})$ is just the closed subscheme of X determined by $\mathfrak{a}$. □

Now, we can give a much quicker proof of the characterization of closed subschemes of affine schemes (recall Theorem 2.85):

Corollary 2.107.1 (Closed Subschemes of Affine Schemes)**.** *If $X = \operatorname{Spec} A$ is an affine scheme, there is a bijective correspondence between ideals $\mathfrak{a}$ in A and closed subschemes Y of X, given by $\mathfrak{a} \mapsto \operatorname{Spec} A/\mathfrak{a} \subseteq X$. In particular, every closed subscheme of an affine scheme is affine.*

Proof. This follows from the above result and Corollary 2.103.1. □

We conclude with some more technical results about sheaf modules:

Definition 2.108 (Dual)**.** Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathscr{E}$ be an $\mathcal{O}_X$-module. We define the *dual* of $\mathscr{E}$, denoted $\check{\mathscr{E}}$, to be the sheaf $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathcal{O}_X)$.

First, let us discuss some basic results analogous to results about modules. For this, we need an algebraic lemma:

Lemma 2.109. *Suppose that A is a commutative ring and M, N are A-modules. Define a map $\Phi : \operatorname{Hom}(M, A) \times \operatorname{Hom}(N, A) \to \operatorname{Hom}(M \otimes N, A)$ as follows. Suppose that $(\phi, \psi) \in \operatorname{Hom}(M, A) \times \operatorname{Hom}(N, A)$. This induces a unique natural map $\rho : M \times N \to A$, which is bilinear and hence induces a map $M \otimes N \to A$, which we define to be $\Phi(\phi, \psi)$. Then Φ is an isomorphism.*

Proof. This fact follows because there is an inverse map Ψ to Φ. Namely, suppose we are given a map $\rho : M \otimes N \to A$, we may compose this map with the natural map $M \times N \to M \otimes N$ to get a map $M \times N \to A$. Then, we may compose these maps with the natural maps $M \to M \times N$ and $N \to M \times N$ to get a pair $\Psi(\rho)$ of maps $M \to A$, $N \to A$. It is easy to check that $\Phi(\Psi(\rho)) = \rho$ and $\Psi(\Phi(\phi, \psi)) = (\phi, \psi)$, so the result follows. □

Theorem 2.110.

(a) Let $\mathscr{E}$ be a locally free $\mathcal{O}_X$-module of finite rank. Then $(\check{\mathscr{E}})\check{} \simeq \mathscr{E}$.

(b) Let $\mathscr{E}$ be a locally free $\mathcal{O}_X$-module of finite rank. Then, for any $\mathcal{O}_X$-module $\mathscr{F}$, $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{F}) \simeq \check{\mathscr{E}} \otimes_{\mathcal{O}_X} \mathscr{F}$.

(c) $(\check{\mathscr{E}} \otimes \check{\mathscr{F}}) \simeq (\mathscr{E} \otimes \mathscr{F})\check{}$.

(d) For any $\mathcal{O}_X$-modules $\mathscr{F}, \mathscr{G}$, $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{E} \otimes \mathscr{F}, \mathscr{G}) \simeq \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \operatorname{Hom}_{\mathcal{O}_X}(\mathscr{E}, \mathscr{G}))$.

(e) (Projection Formula). If $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces, if $\mathscr{F}$ is an $\mathcal{O}_X$-module, and if $\mathscr{E}$ is a locally free $\mathcal{O}_Y$-module of finite rank, then there is a natural isomorphism $f_*(\mathscr{F} \otimes_{\mathcal{O}_X} f^*\mathscr{E}) \simeq f_*(\mathscr{F}) \otimes_{\mathcal{O}_Y} \mathscr{E}$.

Proof.
(a): First, we will define a map $\phi : \mathscr{E} \to (\check{\mathscr{E}})\check{}$ as follows: for any open $U \subseteq X$, take $s \in \mathscr{E}(U)$. Then define $\phi_U(s) \in (\check{\mathscr{E}})\check{}(U)$ to be the map taking $\psi \in \operatorname{Hom}_{\mathcal{O}_X}(\mathscr{E}, \mathcal{O}_X)(U)$ to $\psi(s) \in \mathcal{O}_X(U)$.

This is clearly a map of $\mathcal{O}_X$-modules; it is plainly a morphism of sheaves with the natural restriction maps (because the map from a module to its double dual is natural), and furthermore the map $\psi \mapsto \psi(s)$ is a homomorphism of $\mathcal{O}_X(U)$-module. Furthermore, because $\mathscr{E}$ is locally free, we may cover X with opens V_i such that $\mathscr{E}|_{V_i} \simeq \mathcal{O}_{V_i}^n$ for some n. Yet, on these opens, $\phi|_{V_i}$ is an isomorphism, so by uniqueness of gluing, ϕ is an isomorphism, as desired.

(b): First, we define a map $\phi : \check{\mathscr{E}} \otimes_{\mathcal{O}_X} \mathscr{F} \to \operatorname{Hom}_{\mathcal{O}_X}(\mathscr{E}, \mathscr{F})$. By the universal property of the tensor product of sheaves, this is equivalent to finding a $\mathcal{O}_X$-bilinear map $\phi' : \check{\mathscr{E}} \times \mathscr{F} \to \operatorname{Hom}_{\mathcal{O}_X}(\mathscr{E}, \mathscr{F})$. Define $\phi'(U)$ as follows: given $(f, s) \in \check{\mathscr{E}}(U) \times \mathscr{F}(U)$, consider f as a map $\mathscr{E}|_U \to \mathcal{O}_X|_U$. Then, notice that with $s \in \mathscr{F}(U)$, f induces a natural map $\mathscr{E}|_U \to \mathscr{F}|_U$. Such a map is simply an element of $\operatorname{Hom}_{\mathcal{O}_X}(\mathscr{E}, \mathscr{F})(U)$, as desired.

Now, we may again apply the trick of restricting ϕ to an open cover $\{V_i\}$ where $\mathscr{E}|_{V_i}$ is free (and therefore where the restriction of ϕ is an isomorphism), and glue together to see that ϕ is an isomorphism.

(c): First, we will define a bilinear map $\check{\mathscr{E}} \times \check{\mathscr{F}} \to (\mathscr{E} \otimes \mathscr{F})\check{}$. For this, suppose we are given an element $(\phi, \psi) \in \check{\mathscr{E}} \times \check{\mathscr{F}}$; that is, $\phi : \mathscr{E} \to \mathcal{O}_X$ and $\psi : \mathscr{F} \to \mathcal{O}_X$ are morphisms. Then we have a unique map $\rho : \mathscr{E} \times \mathscr{F} \to \mathcal{O}_X$ commuting with the projections onto $\mathscr{E}, \mathscr{F}$ and ϕ, ψ. Now, ρ is plainly a bilinear map, so it induces a map $(\mathscr{E} \otimes \mathscr{F})\check{} \to \mathcal{O}_X$. This provides a map $\Phi : \check{\mathscr{E}} \times \check{\mathscr{F}} \to (\mathscr{E} \otimes \mathscr{F})\check{}$. This map is natural by the uniqueness of ρ, which itself is guaranteed by the uniqueness of the induced map in the universal property of the (co)product. Therefore, it suffices to prove that Φ is an isomorphism. But this is easy: we may simply check that Φ_U is an isomorphism for each U using Lemma 2.109.

(d): Recall that for each open $U \subseteq X$ there is a *natural* isomorphism $\operatorname{Hom}_{\mathcal{O}_X(U)}(\mathscr{E}(U) \otimes \mathscr{F}(U), \mathscr{G}(U)) \simeq \operatorname{Hom}_{\mathcal{O}_X(U)}(\mathscr{F}(U), \operatorname{Hom}_{\mathcal{O}_X(U)}(\mathscr{E}(U), \mathscr{G})(U))$ afforded by the tensor-hom adjunction. Because this isomorphism is $\mathcal{O}_X(U)$-linear, and natural in $\mathscr{F}(U)$ and $\mathscr{G}(U)$, the components of the isomorphism may be glued together to give an $\mathcal{O}_X$-isomorphism $\operatorname{Hom}_{\mathcal{O}_X}(\mathscr{E} \otimes \mathscr{F}, \mathscr{G}) \simeq \operatorname{Hom}_{\mathcal{O}_X}(\mathscr{F}, \operatorname{Hom}_{\mathcal{O}_X}(\mathscr{E}, \mathscr{G}))$. Indeed, they may be glued together to give a $\mathcal{O}_X|_U$-isomorphism $\operatorname{Hom}_{\mathcal{O}_X}(\mathscr{E} \otimes \mathscr{F}, \mathscr{G})|_U \simeq \operatorname{Hom}_{\mathcal{O}_X}(\mathscr{F}, \operatorname{Hom}_{\mathcal{O}_X}(\mathscr{E}, \mathscr{G}))|_U$ for each U. This isomorphisms are natural (by the naturality of the isomorphism of the tensor-hom adjunction), so they are the components of an isomorphism $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{E} \otimes \mathscr{F}, \mathscr{G}) \simeq \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \operatorname{Hom}_{\mathcal{O}_X}(\mathscr{E}, \mathscr{G}))$. This also inherits naturality from the naturality of the isomorphism in the tensor-hom adjunction.

(e): By parts (a) and (b), we have

$$f_* \mathscr{F} \otimes_{\mathcal{O}_Y} \mathscr{E} \simeq f_* \mathscr{F} \otimes_{\mathcal{O}_Y} (\check{\mathscr{E}})\check{} \simeq \mathscr{H}om_{\mathcal{O}_Y}(\check{\mathscr{E}}, f_* \mathscr{F}).$$

Furthermore, for any $\mathscr{F}$ and $\mathscr{G}$, we have $\operatorname{Hom}_{\mathcal{O}_X}(f^* \mathscr{G}, \mathscr{F}) \simeq \operatorname{Hom}_{\mathcal{O}_Y}(\mathscr{G}, f_* \mathscr{F})$, and also the corresponding statement for each restriction, so we glue together an $\mathcal{O}_Y$-isomorphism $f_* \mathscr{H}om_{\mathcal{O}_X}(f^* \mathscr{G}, \mathscr{F}) \simeq \mathscr{H}om_{\mathcal{O}_Y}(\mathscr{G}, f_* \mathscr{F})$. Plugging this result into the above,

$$f_* \mathscr{F} \simeq \mathscr{H}om_{\mathcal{O}_Y}(\check{\mathscr{E}}, f_* \mathscr{F}) \simeq f_* \operatorname{Hom}_{\mathcal{O}_X}(f^*(\check{\mathscr{E}}), \mathscr{F}) \simeq f_*(\mathscr{F} \otimes_{\mathcal{O}_X} f^*(\check{\mathscr{E}})\check{}) \simeq f_*(\mathscr{F} \otimes_{\mathcal{O}_X} f^*(\mathscr{E}))$$

and we are done. ☐

Next, let us discuss a characterization of "invertible sheaves" which explains the terminology. For this, we need an algebraic lemma.

Lemma 2.111. *Suppose that $A, \mathfrak{m}$ is a local ring with residue field k and finitely-generated A-modules M, N. Then if $M \otimes_A N \simeq A$, $M \simeq A$ and $N \simeq A$.*

Proof. First, notice that

$$k \simeq k \otimes_A A \simeq k \otimes_A (M \otimes_A N) \simeq k \otimes_A (M \otimes_A N) \otimes_k k \simeq (k \otimes_A M) \otimes_k (k \otimes_A N)$$

so both $(k \otimes_A M) \simeq M/\mathfrak{m}M$ and $(k \otimes_A N) \simeq N/\mathfrak{m}N$ are 1-dimensional k-vector spaces. Then any nonzero elements of $(k \otimes_A M)$ and $(k \otimes_A N)$ generate M and N over A by Nakayama's Lemma, so M and N have rank 1. Furthermore, they are free of rank 1, because any element which annihilates M or N annihilates $M \otimes_A N \simeq A$, and the only annihilator of A is 0. Hence $M \simeq A$ and $N \simeq A$, as desired. ☐

Theorem 2.112 (Alternate Characterization of Invertible Sheaves)**.** *Let X be a Noetherian scheme, and let $\mathscr{F}$ be a coherent sheaf.*

(a) If the stalk $\mathscr{F}_x$ is a free $\mathcal{O}_x$-module for some point $x \in X$, then there is a neighborhood U of x such that $\mathscr{F}|_U$ is free.

(b) $\mathscr{F}$ is locally free if and only if its stalks $\mathscr{F}_x$ are free $\mathcal{O}_x$-modules for all $x \in X$.

(c) $\mathscr{F}$ is invertible (i.e., locally free of rank 1) if and only if there is a coherent sheaf $\mathscr{G}$ such that $\mathscr{F} \otimes \mathscr{G} \simeq \mathcal{O}_X$. (This justifies the terminology invertible: it means that $\mathscr{F}$ is an invertible element of the monoid of coherent sheaves under the operation $\otimes$.)

Proof.
(a): Let V be an open affine neighborhood of x with $V = \operatorname{Spec} A$, and let $\mathfrak{p} \lhd_{\text{pr}} A$ be the prime ideal corresponding to x. By Proposition 5.4, $\mathscr{F}|_V = \widetilde{M}$ for some finite A-module M. The statement "$\mathscr{F}_x$ is a free $\mathcal{O}_x$-module" then becomes "$M_\mathfrak{p}$ is a finite free $A_\mathfrak{p}$-module". Now, suppose that $m_1/a_1, \dots, m_n/a_n$ freely generate $M_\mathfrak{p}$ as an $A_\mathfrak{p}$-module. Then also $m_1, \dots, m_n$ freely generate $M_\mathfrak{p}$ as an $A_\mathfrak{p}$-module, since $a_1, \dots, a_n \in A \setminus \mathfrak{p}$ are units. This induces a map $\phi : A^n \to M$ given by sending the ith basis element to m_i.

Now, ϕ has an associated morphism Φ of sheaf modules. Let the support of the kernel of Φ be called K, and the support of the cokernel of Φ be called C. Now, since Φ is an isomorphism at x by assumption, x is not in K or C. Recall that the support of the kernel and the cokernel are closed, so K and C are closed; in particular $K \cup C$ is closed whence $X \setminus K \cup C$ is open. Therefore exists some open neighborhood $D(f)$ of x contained in $X \setminus K \cup C$. Yet on $D(f)$, Φ is an isomorphism, since the kernel and cokernel are zero everywhere on $D(f)$. Hence $\phi_f : A_f^n \to M_f$ is an isomorphism and $\mathscr{F}|_{D(f)}$ is free.

(b): If $\mathcal{F}$ is locally free, then obviously its stalks $\mathcal{F}_x$ are free $\mathcal{O}_x$-modules for all $x \in X$. The other direction follows immediately from (a). To see why, suppose that all the stalks $\mathcal{F}_x$ of $\mathcal{F}$ are free $\mathcal{O}_x$-modules. Then for each point $x \in X$ there exists a neighborhood U_x of x such that $\mathcal{F}|_{U_x}$ is free. But then $\{U_x\}_{x \in X}$ is an open cover of X such that $\mathcal{F}|_{U_x}$ is free for each U_x, so $\mathcal{F}$ is locally free.

(c): Suppose $\mathscr{F}$ is invertible. Then by 5.1(b), $\mathscr{F} \otimes_{\mathcal{O}_X} \check{\mathscr{F}} \simeq \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{F})$. Yet we may glue together a isomorphism $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{F}) \simeq \mathcal{O}_X$ by restricting to an open cover where the restriction of $\mathscr{F}$ is free of rank 1 (clearly, for such restrictions, the isomorphism holds), so in fact $\mathscr{F} \otimes_{\mathcal{O}_X} \check{\mathscr{F}} \simeq \mathcal{O}_X$, as desired.

Conversely, suppose that there exists a coherent sheaf $\mathscr{G}$ such that $\mathscr{F} \otimes \mathscr{G} \simeq \mathcal{O}_X$. Then, $(\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G})_x = \mathscr{F}_x \otimes_{\mathcal{O}_X} \mathscr{G}_x = \mathcal{O}_x$ for all $x \in X$, so by part (a) we have reduced to Lemma 2.111. ☐

Theorem 2.113 (A Criterion for Being Locally Free)**.** *Let X be a Noetherian scheme, and $\mathscr{F}$ a coherent sheaf on X. Consider the function*

$$\varphi(x) = \dim_{k(x)} \mathscr{F}_x \otimes_{\mathcal{O}_x} k(x),$$

where $k(x) = \mathcal{O}_x/\mathfrak{m}_x$ is the residue field at the point x. Then,

(a) The function φ is upper semi-continuous, i.e., for any $n \in \mathbb{Z}$, the set $\{x \in X \mid \varphi(x) \geq n\}$ is closed.

(b) If $\mathscr{F}$ is locally free, and X is connected, then φ is a constant function.

(c) Conversely, if X is reduced, and φ is constant, then $\mathscr{F}$ is locally free.

Solution 2.1.
(a): By Lemma 6.12, it suffices to show the result when X is affine, say equal to $\operatorname{Spec} A$ for a Noetherian ring A. Since $\mathscr{F}$ is a coherent sheaf, it is equal to $\widetilde{M}$ for some finite A-module M, say generated by nonzero elements $m_1, \dots, m_k$. Now, to show that $\{\mathfrak{p} \in X \mid \varphi(\mathfrak{p}) \geq n\}$ is closed for each n, it suffices to show that $\{\mathfrak{p} \in X \mid \varphi(\mathfrak{p}) \leq n\}$ is open for each n. For this, it suffices to show if $\mathfrak{p} \in X$ satisfies $\varphi(\mathfrak{p}) = n$, there is a neighborhood U of $\mathfrak{p}$ with $\varphi(\mathfrak{q}) \leq n$ for all $\mathfrak{q} \in U$.

Now, let $\mathfrak{p} \lhd_{\text{pr}} A$, so that $\varphi(\mathfrak{p}) = \dim_{k(\mathfrak{p})} M_\mathfrak{p} \otimes_{A_\mathfrak{p}} k(\mathfrak{p}) = \dim_{k(\mathfrak{p})} M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$. Choose a $k(\mathfrak{p})$-basis $v_1', \dots, v_n'$ for this vector space. By Nakayama's Lemma, these elements lift to a spanning set $v_1, \dots, v_n$ of $M_\mathfrak{p}$ as an $A_\mathfrak{p}$-module. Indeed, we may assume that $v_1, \dots, v_n$ lie in M, by clearing denominators if necessary.

Now, since $v_1, \dots, v_n$ span $M_\mathfrak{p}$, there exist $a_{ij} \in M_\mathfrak{p}$ such that $m_i = \sum_j a_{ij} v_j$ for each i. Write $a_{ij} = \frac{b_{ij}}{c_{ij}}$ for $b_{ij} \in A$ and $c_{ij} \in A \setminus \mathfrak{p}$. Then, let $c \in A$ be the product of all the c_{ij}. Then plainly $\mathfrak{p} \in D(c)$. Indeed, I claim that $D(c)$ is the desired neighborhood U of $\mathfrak{p}$. For this, suppose that $\mathfrak{q} \in D(c)$. Then a_{ij} is an element of $M_\mathfrak{q}$, since we can write it as $\frac{a_{ij}c}{c}$ (note that $a_{ij}c \in A$ and $c \in A \setminus \mathfrak{q}$ since $\mathfrak{q} \in D(c)$). Hence $m_i = \sum_j a_{ij} v_j$ is a valid expression in $M_\mathfrak{q}$. Yet then the m_i generate $M_\mathfrak{q}$ as a $A_\mathfrak{q}$-module, so $v_1, \dots, v_n$ generate $M_\mathfrak{q}$ as an $A_\mathfrak{q}$-module and $\varphi(\mathfrak{q}) \le n$, as desired.

(b): If $\mathscr{F}$ is locally free, then there is an open cover $\{U_i\}$ such that $\mathscr{F}|_{U_i} \simeq \mathcal{O}_X|_{U_i}^{\oplus r_i}$ for some r_i. But then $\mathscr{F}_x \otimes_{\mathcal{O}_x} k(x) \simeq \mathcal{O}_x^{\oplus r_i} \otimes_{\mathcal{O}_x} k(x) \simeq k(x)^{\oplus r_i}$ is a r_i-dimensional $k(x)$-vector space for all $x \in U_i$, so φ is constant equal to r_i on U_i. Since the U_i cover X, and X is connected, this implies that φ is constant on X.

(c): Plainly it suffices to consider the affine case, since being locally free is a local property. Therefore, assume that $X = \operatorname{Spec} A$ for a reduced Noetherian ring A. Then, since it is coherent, $\mathscr{F} = \widetilde{M}$ for some finite A-module M. Now, by Theorem 2.112, it suffices to argue that $\mathscr{F}_\mathfrak{p}$ is a free $A_\mathfrak{p}$-module for each $\mathfrak{p} \lhd_{\text{pr}} A$. Now, since φ is constant, say equal to n everywhere, $M_\mathfrak{p} \otimes_{A_\mathfrak{p}} k(\mathfrak{p}) \simeq M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$ is a n-dimensional $k(\mathfrak{p})$-vector space. Take a basis $v_1', \dots, v_n'$ of this vector space; by Nakayama's Lemma these elements lift to a spanning set $v_1, \dots, v_n$ of $M_\mathfrak{p}$. It remains to demonstrate that $v_1, \dots, v_n$ are linearly independent over $A_\mathfrak{p}$.

For this, take a linear sum $\sum a_i v_i = 0$ with $a_i \in A_\mathfrak{p}$. Now, because the v_i are a basis for $M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$, the image of the a_i must be zero; that is, $a_i \in \mathfrak{p}$ for each i. Now take $\mathfrak{q} \subseteq \mathfrak{p}$; the images of $v_1, \dots, v_n$ in $M_\mathfrak{q}$ generate $M_\mathfrak{q}$, and we still have $\sum a_i v_i = 0$ in $M_\mathfrak{q}$. Therefore, $v_1, \dots, v_n$ span the n-dimensional $k(\mathfrak{q})$-vector space $M_\mathfrak{q}/\mathfrak{q}M_\mathfrak{q}$, so they must be linearly independent in $M_\mathfrak{q}/\mathfrak{q}M_\mathfrak{q}$. Yet this forces the image of the a_i to be zero, whence $a_i \in \mathfrak{q}$ for each i. In summary, $a_i \in \mathfrak{q}$ for any $\mathfrak{q} \subseteq \mathfrak{p}$.

Yet this implies that $a_i \in \bigcap_{\mathfrak{q} \subseteq \mathfrak{p}} \mathfrak{q}$, which is the nilradical of $A_\mathfrak{p}$. Yet the localization of any reduced ring is reduced, so $A_\mathfrak{p}$ is reduced and therefore has zero nilradical. Hence $a_i = 0 \in A_\mathfrak{p}$, as desired.

2.7 Recontextualizing Varieties as Schemes

First, using our currently exist definition of varieties, let us discuss how varieties can be interpreted as schemes using a fully faithful functor.

Theorem 2.114. *Let k be an algebraically closed field. There is a natural fully faithful functor $t : \mathfrak{Var}(k) \to \mathfrak{Sch}(k)$ from the category of varieties over k to the category of schemes over k. For any variety V, its topological space is homeomorphic to the set of closed points of $\operatorname{sp}(t(V))$, and its sheaf of regular functions is obtained by restricting the structure sheaf of $t(V)$ via this homeomorphism.*

Proof. Pg. 78-79 of Hartshorne. □

However, this inspires us to make a purely scheme-theoretic definition of varieties. This new definition will in fact be a generalization of our old definition of varieties, which we will call *quasi-projective varieties.* To distinguish them for the old definition, we call these "abstract varieties".

Definition 2.115 (Abstract Variety)**.** An *abstract variety* (or, from here on out, just *variety*) is an integral separated scheme of finite type over an algebraically closed field k. If it is proper over k, we will also say it is *complete*. Intuitively, an abstract variety is one which locally looks like affine varieties, just as a scheme locally looks like affine schemes.

3 Curves

To be completed in the summer.

4 Computational Algebraic Geometry

To be completed in the summer.

5 Arithmetic Geometry

To be completed in the summer.

Appendix

A Results from Commutative Algebra

A.1 Assorted Useful Facts

Lemma 6.1. *Let A be a reduced commutative ring of finite Krull dimension. Then A has a unique minimal prime ideal $\mathfrak{p}$ if and only if A is an integral domain.*

Proof. ($\Rightarrow$) Suppose $\mathfrak{p}$ is the unique minimal prime ideal of A. First, recall that the intersection of all prime ideals of A is the nilradical of A; in our case, this implies that the intersection of all prime ideals of A is (0). Yet any prime ideal $\mathfrak{q}$ of A contains $\mathfrak{p}$, and hence $\mathfrak{p} \subseteq \bigcap_{\mathfrak{q} \lhd_{\text{pr}} A} \mathfrak{q}$. To see why, suppose that $\mathfrak{q}$ does not contain $\mathfrak{p}$. Then, since $\mathfrak{q}$ is not equal to $\mathfrak{p}$, it is not minimal, so it must properly contain some $\mathfrak{q}_1 \lhd_{\text{pr}} A$, which cannot be equal to $\mathfrak{p}$ since then $\mathfrak{q}$ would contain $\mathfrak{p}$. But we can repeat this process to choose an infinite chain $\mathfrak{q} \supsetneq \mathfrak{q}_1 \supsetneq \mathfrak{q}_2 \supsetneq \cdots$ which contradicts the fact that A has finite dimension. Hence $\mathfrak{p} \subseteq (0)$, implying that $\mathfrak{p} = (0)$ is prime and hence that A is an integral domain, as desired.

($\Leftarrow$) Suppose that A is an integral domain. Then (0) is a prime ideal of A contained in every other (prime) ideal of A, hence it is the unique minimal prime ideal. ☐

A.2 Hilbert's Nullstellensatz

Our proof of Noether Normalization, which is used to prove the Nullstellensatz, is adapted from here.

First, we begin with a technical lemma which outlines a useful automorphism of polynomial rings:

Lemma 6.2. *Suppose that k is a field and $f \in k[x_1, \ldots, x_n]$ is a nonzero polynomial in n variables over f. Let N be an integer greater than $\deg(f)$. Now define $\phi : k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]$ to be the k-algebra automorphism given as follows:*

$$x_1 \mapsto x_1 + x_n^N \qquad x_2 \mapsto x_2 + x_n^{N^2} \qquad \cdots \qquad x_{n-1} \mapsto x_{n-1} + x_n^{N^{n-1}} \qquad x_n \mapsto x_n.$$

Then $\phi(f)$ is equal to a nonzero scalar of k times a polynomial g which is monic in x_n when considered as a polynomial in one variable over $k[x_1, \ldots, x_{n-1}]$. That is, the term of $\phi(f)$ in which x_n appears to the highest power has the form cx_n^m for some $c \in k^\times$.

Proof. First simplify f by combining like terms. Then, consider any nonzero monomial $cx_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$ in f (note that $c \in k^\times$). Then the image of this monomial under ϕ is

$$\phi(cx_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}) = (x_1 + x_n^N)^{a_1}(x_2 + a_n^{N^2})_2^a \cdots (x_{n-1} + x_n^{N^{n-1}})x_n^{a_n}.$$

When expanded, the term of this polynomial in which x_n appears to the highest power is plainly equal to $c(x_n^N)^{a_1}(x_n^{N^2})^{a_2}\cdots(x_n^{N^{n-1}})^{a_{n-1}}x_n^{a_n} = cx_n^{a_n + a_1N + a_2N^2 + \cdots + a_{n-1}N^{n-1}} = cx_n^m$.

Now, $N > \deg(f)$ implies $N > a_1, \ldots, a_n$. Now take any two distinct monomials f_1, f_2 of f (recall that we have combined like terms, so the powers of $x_1, \ldots, x_n$ cannot be the same in both monomials). Since any integer has a unique base N representation, the terms of $\phi(f_1)$ and $\phi(f_2)$ in which x_n appears to the highest power must have different degree. In other words, there is a unique nonzero monomial of f whose image has the term with the strictly greatest power of x_n, and said term has the form cx_n^m for some $c \in k^\times$. ☐

Lemma 6.3 (Noether Normalization Lemma)**.** *Let k be a field and A a finitely-generated k-algebra. Then, there are elements $z_1, \ldots, z_m \in A$ such that $z_1, \ldots, z_m$ are algebraically independent over k, and A is finite (in particular integral) over $k[z_1, \ldots, z_m]$.*

Proof. We will use induction on the number n of generators of A over k. Now, in the base case $n = 0$, $A = k$ and the result is trivial. For the inductive step, suppose that $n > 0$ and that the result holds whenever the number of generators is less than n. Let $y_1, \ldots, y_n$ generate A over k. If the y_i are algebraically independent over k, then we may assign $z_i = y_i$ and we are done.

On the other hand, suppose that the y_i are not algebraically independent over k. Then there is a nonzero polynomial $f \in k[x_1, \dots, x_n]$ such that $f(y_1, \dots, y_n) = 0$. Now, define $y_1' = y_1 - y_n^N, \dots, y_{n-1}' = y_{n-1} - y_n^{N^{n-1}}$, and $y_n' = y_n$, where $N > \deg(f)$; these elements also generate A over k. Now, recalling how ϕ was defined in Lemma 6.2, notice that $y_1, \dots, y_n$ satisfy the polynomial $\phi(f) = g$. By Lemma 6.2, by replacing g by $c^{-1}g$, we may assume that g is monic in x_n with coefficients in $k[x_1, \dots, x_{n-1}]$. Therefore y_n' is integral over $k[x_1, \dots, x_{n-1}]$, so $A = k[y_1', \dots, y_n']$ is a finite $k[y_1', \dots, y_{n-1}']$-module. But then, by the inductive hypothesis, there exist algebraically independent $z_1, \dots, z_m \in k[y_1', \dots, y_{n-1}']$ such that $k[y_1', \dots, y_{n-1}']$ is a finite $k[z_1, \dots, z_m]$-module. But then A is a finite $k[z_1, \dots, z_m]$-module, so we are done. □

Proposition 6.4 (Integral Extension of Integral Domains)**.** *Let $A \subseteq B$ be an integral extension of integral domains. Then A is a field if and only if B is a field.*

Proof. Assume that A is a field and take $0 \neq x \in B$. Then there is a monic relation $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ with $a_i \in A$. We may assume that $a_0 \neq 0$. Now, A is a field, therefore

$$x^{-1} = -a_0^{-1}(x^{n-1} + a_{n-1}x^{n-2} + \cdots + a_2 x + a_1) \in B$$

so B is a field. Similarly, assume that B is a field and $0 \neq x \in A$. Then $x^{-1} \in B$, so x^{-1} is integral over A. Then there is a relation of the form

$$x^{-n} + a_{n-1}x^{n-1} + \cdots + a_0 = 0$$

whence $x^{-1} = x^{-1} = -a_{n-1} - a_{n-2}x - \cdots - x^{n-1}a_0 \in A$, so A is indeed a field, as desired. □

Lemma 6.5 (Zariski's Lemma)**.** *If L/k be a field extension such that L is a finitely-generated k-algebra. Then L/k is a finite field extension.*

Proof. By Noether's Normalization Lemma (Lemma 6.3), there exists an injective k-algebra morphism $\phi : k[z_1, \dots, z_r] \hookrightarrow L$. In particular, L is finite over $k[z_1, \dots, z_r]$, so it is integral over $k[z_1, \dots, z_r]$. By Proposition 6.4, since L and $k[z_1, \dots, z_r]$ are both integral domains, L is a field if and only if $k[z_1, \dots, z_r]$ is a field. Yet $k[z_1, \dots, z_r]$ is a field if and only if $r = 0$, so L is finite over k and we are done. □

Proposition 6.6 (The Weak Nullstellensatz)**.** *Maximal ideals of $k[x_1, \dots, x_n]$ correspond precisely to points of $\mathbb{A}_k^n$. More precisely, every maximal ideal of $A = k[x_1, \dots, x_n]$ is of the form*

$$\mathfrak{m} = (x_1 - a_1, \dots, x_n - a_n) \text{ for some } a_1, \dots, a_n \in k$$

and every ideal of the form $(x_1 - a_1, \dots, x_n - a_n)$ for some $a_1, \dots, a_n \in k$ is maximal.

Proof. Clearly every ideal of the form $\mathfrak{a} = (x_1 - a_1, \dots, x_n - a_n)$ for some $a_1, \dots, a_n \in k$ is maximal, since $k[x_1, \dots, x_n]/\mathfrak{a} \simeq k$, a field (this isomorphism is obvious since taking said quotient is like "replacing x_i with a_i"). On the other hand, $k[x_1, \dots, x_n]/\mathfrak{m}$ is a finitely generated k-algebra for any ideal $\mathfrak{m} \lhd k[x_1, \dots, x_n]$. If, furthermore, $\mathfrak{m}$ is maximal, $K = k[x_1, \dots, x_n]/\mathfrak{m}$ is a field, so by Zariski's Lemma (see Lemma 6.5), K is a finite extension over k. Yet k is algebraically closed, so $K = k$.

By taking a_i to be the image of x_i for each $i \in \{1, \dots, n\}$, we see that $(x_1 - a_1, \dots, x_n - a_n) \subseteq \mathfrak{m}$. Yet we already know that $(x_1 - a_1, \dots, x_n - a_n)$ is maximal, so $\mathfrak{m} = (x_1 - a_1, \dots, x_n - a_n)$, as desired. □

Corollary 6.6.1. *If $J \lhd k[x_1, \dots, x_n]$, then J is equal to $k[x_1, \dots, x_n]$ if and only if $V(J) = \varnothing$.*

Proof. If J is a proper ideal, it is contained in a maximal ideal $\mathfrak{m}$. Then $V(\mathfrak{m}) \subseteq V(J)$, but $V(\mathfrak{m})$ is a single point (by the Weak Nullstellensatz) so $V(J)$ is nonempty. On the other hand, if J is not a proper ideal (it is the whole ring), then $V(J) = \varnothing$ since the polynomial 1 vanishes nowhere. □

Theorem 6.7 (Hilbert's Nullstellensatz)**.** *For any $\mathfrak{a} \lhd k[x_1, \dots, x_n]$, $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.*

Proof. Clearly, $\sqrt{\mathfrak{a}} \subseteq I(Z(\mathfrak{a}))$. Thus it suffices to show that $I(Z(\mathfrak{a})) \subseteq \sqrt{\mathfrak{a}}$. Take $g \in I(Z(\mathfrak{a}))$; we will show that $g^j \in \mathfrak{a}$ for some $j \in \mathbb{N}$. To do this, we will use Rabinowitsch's trick, which goes as follows.

Let $f_1, \ldots, f_m$ generate $\mathfrak{a}$. Then $f_1, \ldots, f_m, x_{n+1}g - 1 \in k[x_1, \ldots, x_{n+1}]$ have no common zeros in $\mathbb{A}_k^{n+1}$. This is because, in $Z(\mathfrak{a})$, the former polynomials are all 0, but the last polynomial is -1. But all of $f_1, \ldots, f_m$ cannot vanish outside $Z(\mathfrak{a})$ (by definition), so there are no common zeros outside of $Z(\mathfrak{a})$ either. Thus, by Corollary 6.6.1, these polynomials generate the entirety of $k[x_1, \ldots, x_{n+1}]$. In particular:

$$1 = p_1 f_1 + \cdots + p_m f_m + p_{m+1}(x_{n+1}g - 1)$$

for some $p_1, \ldots, p_{m+1} \in k[x_1, \ldots, x_{n+1}]$. Under the homomorphism $k[x_1, \ldots, x_{n+1}] \to k(x_1, \ldots, x_n)$ given by fixing $k[x_1, \ldots, x_n]$ and sending $x_{n+1} \mapsto g^{-1}$, we see that the final term vanishes;

$$1 = p_1(x_1, \ldots, x_n, g^{-1}) f_1 + \cdots + p_m(x_1, \ldots, x_n, g^{-1}) f_m$$

whence, by letting j be the largest power to which g^{-1} appears in any of the f_i, we see that

$$g^j = q_1 f_1 + \cdots + q_m f_m \in J$$

for some $q_1, \ldots, q_m \in k[x_1, \ldots, x_n]$, as desired. □

A.3 Dimension Theory of Noetherian Rings

At some point, I may add proofs to this section instead of relying on references, but for now these will do.

Theorem 6.8. *Let k be a field, and A an integral domain which is a finitely-generated k-algebra. Then the dimension of A is equal to the transcendence degree of the quotient field $\operatorname{Frac} A$ over k, and for any prime ideal $\mathfrak{p} \lhd A$, we have $\operatorname{ht}\mathfrak{p} + \dim A/\mathfrak{p} = \dim A$.*

Proof. This can be found in §14 of Matsumura's *Commutative Algebra* or, when k is algebraically closed, Ch. 11 of Atiyah-Macdonald's *Introduction to Commutative Algebra* □

Theorem 6.9 (Krull's Haupidealsatz)**.** *Let A be a Noetherian ring, and let $a \in A$ be a nonunit non-zero divisor. Then every minimal prime ideal containing a has height 1.*

Proof. This can be found in Ch. 11 of Atiyah-Macdonald's *Introduction to Commutative Algebra.* □

Theorem 6.10. *A Noetherian domain A is a UFD if and only if every prime ideal of height 1 is principal.*

Proof. This can be found in §19 of Matsumura's *Commutative Algebra.* □

B Results from Topology

B.1 Local Conditions

Clearly, being open is a local condition. That is,

Lemma 6.11 (Openness is a Local Condition)**.** *Let X be a top. space with open cover $\{U_i\}$; then $Y \subseteq X$ is open iff $Y \cap U_i$ is open in U_i for each i.*

Proof. If Y is open, then by definition of the subspace topology $Y \cap U_i$ is open for each i. On the other hand, if $Y \cap U_i$ is open in U_i for each i, then $Y \cap U_i$ is open in X, whence $Y = \bigcup_i Y \cap U_i$ is open. □

However, what might be less obvious is that closedness is *also* a local condition.

Lemma 6.12 (Closedness is a Local Condition)**.** *Let X be a top. space with open cover $\{U_i\}$; then $Y \subseteq X$ is closed iff $Y \cap U_i$ is closed in U_i for each i.*

Proof. Suppose that $Y \cap U_i$ is closed in U_i for each i. This is equivalent to $U_i \setminus (Y_i \cap U_i)$ being open in U_i for each i. Yet $U_i \setminus (Y \cap U_i) = (X \setminus Y) \cap U_i$, so we have that $(X \setminus Y) \cap U_i$ is open. But then $(X \setminus Y) \cap U_i = V \cap U_i$ for some open $V \subseteq X$. Yet the latter is the intersection of two open sets of X and hence open, so $(X \setminus Y) \cap U_i$ is open in X. Then $X \setminus Y = \bigcup_i (X \setminus Y) \cap U_i$ is open in X as the union of open sets, so Y is closed. □

B.2 Irreducibility and Noetherian Spaces

This section is filled with facts concerning Definition 1.13 and Definition 1.17.

Proposition 6.13. *A topological space X is irreducible if and only if any two nonempty open sets of X have nonempty intersection. That is, X is irreducible if and only if any nonempty open set of X is dense in X.*

Proof. The first statement follows immediately from the definition by taking complements. For the second, a set $U \subseteq X$ is said to be dense if $\overline{U} = X$. This is equivalent to the statement that the only closed set containing U is X itself, which by taking complements is equivalent to the statement that the only open set not intersecting U is the empty set. Hence the second statement is equivalent to the first and we're done. □

Proposition 6.14. *Suppose that X is irreducible. Then any nonempty open subset of X is irreducible.*

Proof. Suppose that U is a nonempty open subset of X. In light of Proposition 6.13, it suffices to show that any two nonempty open sets in U have nonempty intersection. Yet any (nonempty) open set in U is a (nonempty) open set in X, so any two nonempty open sets contained in U have nonempty intersection by considering them as open subsets of X. □

Proposition 6.15. *Suppose that $Y \subseteq X$ is irreducible. Then $\overline{Y} \subseteq X$ is irreducible.*

Proof. Suppose that $\overline{Y}$ is not irreducible. Then there exist closed sets V_1, V_2 of X such that $(V_1 \cap \overline{Y}) \cup (V_2 \cap \overline{Y}) = \overline{Y}$ but $V_1, V_2 \not\supseteq \overline{Y}$. But then, by definition of the closure, $V_1, V_2 \not\supseteq Y$, whereas

$$(V_1 \cap Y) \cup (V_2 \cap Y) = (V_1 \cap \overline{Y} \cap Y) \cup (V_2 \cap \overline{Y} \cap Y) = ((V_1 \cap \overline{Y}) \cup (V_2 \cap \overline{Y})) \cap Y = \overline{Y} \cap Y = Y.$$

Hence Y is not irreducible. Taking the contrapositive yields the desired result. □

Lemma 6.16. *Suppose that X is a topological space with an open cover of irreducible sets $\{U_i\}$. Suppose further that the intersection of any nonempty U_i, U_j is nonempty. Then X is irreducible.*

Proof. Suppose that V_1 and V_2 are two nonempty open sets in X. Now, there exist i, j such that $U_i \cap V_1$ and $U_j \cap V_2$ are both nonempty (since the U_i cover X). Then, since $U_i \cap V_1$ and $U_i \cap U_j$ are non-empty open subsets of U_i, by irreducibility they have nonempty intersection. But then $U_i \cap U_j \cap V_1$ and $U_j \cap V_2$ are nonempty open subsets of U_j, so by irreducibility they have nonempty intersection. But then V_1 and V_2 have nonempty intersection. Hence X is indeed irreducible. □

Proposition 6.17. *Suppose that X is a topological space which is irreducible and Hausdorff. Then X is the one-point space. In particular, any affine variety which is Hausdorff consists of a single point.*

Proof. Suppose that X has two distinct points x and y. Then by the Hausdorff condition, they have disjoint neighborhoods; yet any two nonempty open sets of an irreducible space are not disjoint, so this is impossible. This contradiction proves that X must be the one-point space. □

Proposition 6.18. *Suppose that X is a topological space which is Noetherian and Hausdorff. Then X is finite and has the discrete topology. In particular, any affine algebraic set which is Hausdorff is a finite collection of points with the discrete topology.*

Proof. By Proposition 1.20, X can be expressed as the finite union of irreducible closed subsets Y_i. Each of these closed subsets are Hausdorff as the subset of a Hausdorff space, so by Proposition 6.17 they are a single point. Hence X is a finite union of closed points (so it also has the discrete topology), as desired. □

Proposition 6.19. *The following conditions are equivalent for a topological space X.*

(i) X is Noetherian; that is, X satisfies the descending chain condition for closed subsets.

(ii) X satisfies the ascending chain condition for open subsets.

(iii) Any nonempty family of closed subsets of X has a minimal element.

(iv) Any nonempty family of open subsets of X has a maximal element.

Proof. We prove this theorem using a cycle of equivalences:

1: *(i)* implies *(iii)*.
Firstly, suppose X is a Noetherian topological space and $\mathcal{F}$ is a nonempty family of closed subsets of X with no minimal element. Take $Y_1 \in \mathcal{F}$. Since $\mathcal{F}$ has no minimal element, there must exist $Y_2 \subsetneq Y_1$. Similarly, for each i, there must exist $Y_{i+1} \subsetneq Y_i$, else Y_i is a minimal element. Therefore, we get a descending chain of closed subsets $Y_1 \supsetneq Y_2 \supsetneq \cdots$ which does not stabilize, a contradiction. Therefore $\mathcal{F}$ cannot exist.

2: *(iii)* implies *(ii)*.
Suppose that X is a topological space such that every nonempty family $\mathcal{F}$ of closed subsets of X has a minimal element. Take an ascending chain $Y_1 \subseteq Y_2 \subseteq \cdots$ of open sets. Define $Z_i = X \setminus Y_i$ for each i; then $\{Z_i\}$ is a family of closed subsets of X. But then by hypothesis this family has a minimal element, say Z_n. Now, $Y_n \subseteq Y_{n+1} \subseteq \cdots$, so $Z_n \supseteq Z_{n+1} \subseteq \cdots$. Hence the minimality condition forces $Z_n = Z_{n+1} = \cdots$, forcing $Y_n = Y_{n+1} = \cdots$, as desired.

3: *(ii)* implies *(iv)*.
Suppose X satisfies the ascending chain condition for open subsets and $\mathcal{F}$ is a nonempty family of open subsets of X which has no maximal element. Take $Y_1 \in \mathcal{F}$. Since $\mathcal{F}$ has no maximal element, there exists $Y_2 \supsetneq Y_1$. Similarly, for each i, there exists $Y_{i+1} \supsetneq Y_i$. Therefore, we get an ascending chain of open subsets $Y_1 \subsetneq Y_2 \subsetneq \cdots$ which does not stabilize, a contradiction. Therefore $\mathcal{F}$ cannot exist.

4: *(iv)* implies *(i)*.
Suppose that X is a topological space such that every nonempty family $\mathcal{F}$ of open subsets of X has a maximal element. Take a descending chain $Y_1 \supseteq Y_2 \supseteq \cdots$ of closed sets. Define $Z_i = X \setminus Y_i$ for each i; then $\{Z_i\}$ is a family of open subsets of X. But then by hypothesis this family has a maximal element, say Z_n. Now, $Y_n \supseteq Y_{n+1} \supseteq \cdots$, so $Z_n \subseteq Z_{n+1} \subseteq \cdots$. But then the maximal forces $Z_n = Z_{n+1} = \cdots$, forcing $Y_n = Y_{n+1} = \cdots$, as desired.

Hence we are done. □

Proposition 6.20. *Any Noetherian space X is* quasicompact *(that is, any open cover has a finite subcover).*

Proof. Let $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ be an open cover of X. Then, form the family

$$\mathcal{F} = \{X \mid X \text{ is a finite union of elements in } \mathcal{U}\}.$$

This family must have a maximal element by Proposition 6.19. This maximal element has the form $U_{\lambda_1} \cup \cdots \cup U_{\lambda_n}$. Furthermore, we must have $U_{\lambda_1} \cup \cdots \cup U_{\lambda_n} = X$, since if there exists $x \in X$ not contained in $U_{\lambda_1} \cup \cdots \cup U_{\lambda_n}$, then we may choose $U_{\lambda_{n+1}}$ covering x (since $\mathcal{U}$ covers all of X) and then $U_{\lambda_1} \cup \cdots \cup U_{\lambda_{n+1}}$ would be an element of $\mathcal{F}$ strictly containing $U_{\lambda_1} \cup \cdots \cup U_{\lambda_n}$, a contradiction. □

Proposition 6.21. *Any subset of a Noetherian space is Noetherian when given the subspace topology.*

Proof. Suppose that X is a Noetherian topological space with subspace Y. Take a descending chain of closed sets $Y_1 \supseteq Y \supseteq \cdots$ in Y. Then, by definition of the subspace topology, for each i there exists X_i (closed in X) such that $X_i \cap Y = Y_i$. This induces a descending chain of closed sets $X_1 \supseteq X_2 \supseteq \cdots$ in X, which stabilizes at X_n by hypothesis. Yet if $X_n = X_{n+1} = \cdots$, then clearly also $Y_n = Y_{n+1} = \cdots$, so the chain in Y stabilizes and we are done. □

Proposition 6.22. *A topological space is noetherian iff every open subset is quasicompact.*

Proof. Suppose that X is a Noetherian topological space. Suppose, for the sake of contradiction, that there exists an open subset U which is not quasicompact. Then there exists an open cover $\{U_i\}_{i \in I}$ of U which has no finite subcover. Define $V_1 = U_{i_1}$ for some i. Then, since $\{U_i\}_{i \in I}$ has no finite subcover, there exists some U_{i_2} such that $U_{i_2} \not\subseteq V_1$. Define $V_2 = V_1 \cup U_{i_2}$, so that $V_2 \supsetneq V_1$. More generally, because $\{U_i\}_{i \in I}$ has no finite subcover, we can find some $U_{i_{n+1}}$ such that $U_{i_{n+1}} \not\subseteq V_n$, and define $V_{n+1} = V_n \cup U_{i_{n+1}}$, so that $V_{n+1} \supsetneq V_n$.

This gives a strictly ascending chain $V_1 \subsetneq V_2 \subsetneq V_3 \subsetneq \cdots$, contradicting the fact that X is Noetherian.

Now suppose that X is a Noetherian topological space. Let U be an open subset of X with finite subcover $\{U_i\}_{i\in I}$. Then let $\mathscr{S}$ be the set of all finite unions of elements in U. Because X is Noetherian, any chain of elements in $\mathscr{S}$ has an upper bound, so by Zorn's Lemma there is a maximal element M in $\mathscr{S}$. But if $M \neq U$, then because the $\{U_i\}$ cover U we may choose U_j such that $M \cup U_j \supsetneq M$, contradicting the maximality of M. Hence we must have $M = U$, so there is a finite subcover of $\{U_i\}_{i\in I}$, so U is quasicompact, as desired. □

B.3 Dimension of Topological Spaces

This section is filled with facts concerning Definition 1.21.

Proposition 6.23. *Suppose that X is a topological space with subset Y. Then $\dim Y \leq \dim X$.*

Proof. Suppose that Y is a subset of a topological space X. Suppose there exists a chain of closed irreducible subsets of Y, say

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n. \tag{1}$$

Now, each Z_i is also irreducible in X by definition. Furthermore, by Proposition 6.15, $\overline{Z_i}$ is also irreducible. Finally, we must show that $\overline{Z_0} \subseteq \overline{Z_1} \subseteq \cdots \subseteq \overline{Z_n}$ is a *strictly* ascending chain. For this, recall (from basic topology, say Munkres' Theorem 17.4 or Theorem 13 in my personal notes here) that if Y is a subspace of X and $S \subseteq Y$, then the closure of S in Y is the intersection of the closure of S in X and Y. In our case, $Z_i = \overline{Z_i} \cap Y$. Hence we cannot have $\overline{Z_i} = \overline{Z_{i+1}}$, since then we would have $Z_i = Z_{i+1}$, a contradiction. Hence $\overline{Z_0} \subsetneq \overline{Z_1} \subsetneq \cdots \subsetneq \overline{Z_n}$ is a strictly ascending chain.

Hence any strictly ascending chain of closed irreducible subsets of Y induces a strictly ascending chain of closed irreducible subsets of X with equal length. Yet this implies that the supremum of all the lengths of such chains in X is bounded below by the supremum of all the lengths of such chains in Y, so $\dim Y \leq \dim X$. □

Proposition 6.24. *If X is a topological space which is covered by a family of open subsets $\{U_i\}$, then $\dim X = \sup \dim U_i$.*

Proof. By Proposition 6.23, plainly we have $\dim X \geq \dim U_i$ for each i. This implies, by properties of the supremum, that $\dim X \geq \sup \dim U_i$. For the other direction, suppose that X has a strict chain of closed irreducible subsets $Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$. Now, Z_0 is nonempty since it is irreducible, so there exists some $x_0 \in Z_0$. x_0 is contained in U for some $U \in \{U_i\}$, by definition of an open cover.

I claim that $Z_0 \cap U \subseteq \cdots \subseteq Z_n \cap U$ is a strict chain of closed irreducible subsets of U. The fact that each $Z_i \cap U$ is closed is a direct consequence of the definition of the subspace topology. Next, we will show that each $Z_i \cap U$ is irreducible. For this, suppose that A, B are proper nonempty closed subsets of U satisfying $A \cup B = Z_i \cap U$. Yet then $\overline{A}$, $\overline{B}$, and $Z_i \setminus U$ are three proper nonempty closed subsets of U whose union is Z_i. But this contradicts the fact that Z_i is irreducible, so indeed $Z_i \cap U$ must be irreducible.

Similarly, the chain is strict; we cannot have $Z_i \cap U = Z_{i+1} \cap U$, since then Z_i and $U \setminus Z_{i+1}$ are closed and nonempty proper subsets of Z_i covering Z_i, which would contradict the fact that Z_i is irreducible. Hence any strict chain of closed irreducible subsets of X induces a strict chain of closed irreducible subsets of some $U \in \{U_i\}$ of the same length, so $\dim X \leq \sup \dim U_i$ and indeed $\dim X = \sup \dim U_i$, as desired. □

Proposition 6.25. *If Y is a closed subset of an irreducible finite-dimensional topological space X such that $\dim Y = \dim X$, then $Y = X$.*

Proof. Suppose, for the sake of contradiction, that X is a finite-dimensional irreducible topological space with a closed subset Y such that $\dim Y = \dim X = n$ but $Y \neq X$. Then there exists a chain of subsets $Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$, irreducible and closed in Y. But each Z_i is plainly irreducible in X. Furthermore, each Z_i is closed in X as the intersection of a closed set in X with the closed set Y.

Yet X is irreducible, closed in itself, and strictly larger than each Z_i (since it is strictly larger than Y and Y contains each Z_i) so we have a strict chain of closed subsets $Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n \subsetneq X$, implying the dimension of $\dim X$ is at least $n+1$, a contradiction. □

B.4 Topology of Affine Schemes

Lemma 6.26. *A closed subset V of a quasicompact topological space X is quasicompact.*

Proof. Let $\{U_i\}$ be an open covering of V. Then by definition of the subspace topology, there exist U_i' such that $U_i = U_i' \cap V$. Then $\{U_i'\}$ and $(X \setminus V)$ form an open covering of X, so there is a finite subcover $(X \setminus V), U_1', \dots, U_n'$ of X. But then $U_1 = U_1' \cap V, \dots, U_n = U_n' \cap V$ is a finite subcover of V, as desired. □

Proposition 6.27. *Suppose that X is a topological space with a cover $U_1, \dots, U_n$ such that U_i is a quasicompact space for each i. Then X is quasicompact.*

Proof. Take an open cover $\{V_j\}$ of X. For each fixed i, $\{V_j \cap U_i\}$ is an open cover of U_i, and therefore it has a finite subcover. This lifts to a finite subset of $\{V_j\}$ covering U_i. But there are only finitely many U_i, and they cover X, so we get a finite subset of $\{V_j\}$ covering X. □

Proposition 6.28. *If X is an affine scheme,* $\mathrm{sp}(X)$ *is quasicompact, but not in general Noetherian.*

Proof. Suppose that $X = \operatorname{Spec} A$ and $\{U_i\}_{i\in I}$ is an open cover of $\mathrm{sp}(X)$. Because the basic affine opens form a basis for the topology on $\mathrm{sp}(X)$, for each $i \in I$, $U_i \supseteq D(a_i)$ for some $a_i \in A$. Now, notice that $\bigcup_{i\in I} D(a_i) = X$ is equivalent to $\bigcap_{i\in I} X \setminus D(a_i) = \bigcap_{i\in I} V((a_i)) = \varnothing$. Yet notice that $\bigcap_{i\in I} V((a_i)) = V\left(\sum_{i\in I}(a_i)\right)$. Yet the only ideal which does not contain any prime ideals is A itself, so we must have $A = \sum_{i\in I}(a_i)$. In particular, by definition there must exist $a_1, \dots, a_n$ such that $1 = a_1 + \cdots + a_n$.

But then $D(a_1), \dots, D(a_n)$ covers X. Indeed, $\bigcap_{j=1}^n V((a_j)) = V((a_1) + \cdots + (a_n)) = V(A) = \varnothing$, so

$$\bigcup_{j=1}^n D(a_i) = X \setminus \bigcap_{j=1}^n (X \setminus D(a_i)) = X \setminus \bigcap_{j=1}^n V((a_i)) = X \setminus \varnothing = X.$$

Yet recall that there exists $U_1 \supseteq D(a_1), \dots, U_n \supseteq D(a_n)$ by our choice of the a_i. Hence $U_1, \dots, U_n$ is the desired finite subcover of $\{U_i\}_{i\in I}$.

Now we will exhibit an example of a non-Noetherian affine scheme. Let $A = k[x_1, x_2, \dots]$ for some field k. Then define $V_n = \bigcap_{i=1}^n V((x_i))$. Clearly, this is an descending chain of closed sets, but I claim that it is *strictly* descending. To see why, notice that $(x_i) \subseteq (x_1, \dots, x_n)$ for $i = 1, \dots, n$, so $(x_1, \dots, x_n) \in V_n$. Yet $(x_{n+1}) \not\subseteq (x_1, \dots, x_n)$, so $(x_1, \dots, x_n) \notin V_{n+1}$. Hence V_{n+1} is a proper subset of V_n, so $V_1, V_2, \dots$ is a strictly descending chain of closed subsets, which proves that $\operatorname{Spec} A$ cannot be Noetherian, as desired. □

Proposition 6.29. *If A is a Noetherian ring, then* $\mathrm{sp}(\operatorname{Spec} A)$ *is a Noetherian topological space.*

Proof. Let $V_1 \supseteq V_2 \supseteq \cdots$ be a descending chain of closed sets. Then there exist ideals $I_1, I_2, \dots$ such that $V_i = V(I_i)$ for each i. Notice that $V(I_i) \supseteq V(I_{i+1})$ implies $I_i \subseteq I_{i+1}$, so by the Noetherian condition there must exist some N such that $I_N = I_{N+1} = \cdots$. But then obviously $V_N = V_{N+1} = \cdots$, as desired. □

Lemma 6.30. *Suppose A is a ring such that* $\operatorname{Spec} A$ *is irreducible. Then the nilradical $\mathcal{N}$ of A is prime.*

Proof. Suppose f and g are two non-nilpotent elements of A. Because the nilradical is precisely the intersection of all prime ideals of A, there exist prime ideals $\mathfrak{p}_1$ not containing f and $\mathfrak{p}_2$ not containing g. Then $\mathfrak{p}_1 \in D((f))$ and $\mathfrak{p}_2 \in D((g))$. Since $D((f))$ and $D((g))$ are nonempty opens, by irreducibility $D((f)) \cap D((g))$ must be nonempty. But $D((f)) \cap D((g)) = D((fg))$, and if $D((fg))$ is nonempty then fg cannot be nilpotent. Hence $f, g \notin \mathcal{N} \Rightarrow fg \notin \mathcal{N}$, so the nilradical is prime. □

B.5 Topology of General Schemes and Zariski Spaces

Definition 6.31 (Generic Point)**.** If X is a topological space, and Z an irreducible closed subset of X, a *generic point* for Z is a point ζ such that $Z = \overline{\{\zeta\}}$.

Definition 6.32 (Zariski Space)**.** A topological space X is a *Zariski space* if it is Noetherian and every (nonempty) closed irreducible subset has a unique generic point.

Definition 6.33 (Specialization and Generization)**.** If x_0, x_1 are points of a topological space X, and if $x_0 \in \overline{\{x_1\}}$, then we say that x_1 *specializes* to x_0, written $x_1 \rightsquigarrow x_0$. We also say x_0 is a *specialization* of x_1, or that x_1 is a *generization* of x_0.

Lemma 6.34. *If $S \subseteq X$ is* stable under generization *(contains every generization of any of its points), then $X \setminus S$ is* stable under specialization *(contains every specialization of any of its points).*

Proof. Suppose $S \subseteq X$ is stable under generization. Now suppose that x_0, x_1 in X are such that $x_0 \in \overline{\{x_1\}}$. By definition, if $x_0 \in S$, then $x_1 \in S$. Then, by contraposition, if $x_1 \notin S$, then $x_0 \notin S$; that is, if $x_1 \in X \setminus S$ is such that $x_0 \in \overline{\{x_1\}}$, then $x_0 \in X \setminus S$. By definition, this means that $X \setminus S$ is stable under specialization. □

Theorem 6.35 (Properties of Zariski Spaces)**.** *Suppose X is a Zariski space. Then:*

(a) Any minimal nonempty closed subset of a Zariski space consists of one point, called a closed point*.*

(b) A Zariski space X satisfies the T_0 axiom: given any two distinct points of X, there is an open set containing one but not the other.

(c) If X is also irreducible, then its generic point is contained in every nonempty open subset of X.

(d) The minimal points, for the partial ordering determined by $x_1 \geq x_0$ if $x_1 \rightsquigarrow x_0$, are the closed points, and the maximal points are the generic points of the irreducible components of X.

(e) Closed subsets are stable under specialization. Similarly, open subsets are stable under generization.

Proof.
(a): Suppose that V is a minimal nonempty closed subset of a Zariski space X; by assumption, there exists a unique generic point $v \in V$. Suppose, for the sake of contradiction, that V is not a singleton. Then, choose any $w \in V \setminus \{v\}$; by uniqueness w is not a generic point so $\overline{\{w\}} \subsetneq V$, contradicting the fact that V is a minimal nonempty closed subset of X. Hence V must be a singleton, as desired.

(b): Let x and y be two distinct points in a Zariski space X. Then $\overline{\{x\}}$ cannot equal $\overline{\{y\}}$, because otherwise $V = \overline{\{x\}}$ would have two generic points, x and y, violating uniqueness. Therefore, we may assume without loss of generality that $\overline{\{x\}} \not\subseteq \overline{\{y\}}$. Then, in particular, $x \notin \overline{\{y\}}$; by definition, this implies that there exists a closed set V containing y but not x. But then $X \setminus V$ is an open set containing x but not y, as desired.

(c): Suppose that X is an irreducible Zariski space. Now, there exists a unique point $x \in X$ with $\overline{\{x\}} = X$. That is, the smallest closed set containing x is X. Now assume that U is a nonempty open set of X not containing x. Then $X \setminus U$ is a closed set containing x which is strictly contained in X, contradicting the fact that the smallest closed set containing x is X. Hence every nonempty open set of X contains x.

(d): Let X be a Zariski space. Then,

(1) The minimal points, for the partial ordering determined by $x_1 > x_0$ if $x_1 \rightsquigarrow x_0$, are the closed points.
Proof: Suppose that x is a closed point. Then x is minimal, as if y is a point with $x \geq y$, then $y \in \overline{\{x\}} = \{x\}$, so $y = x$. On the other hand, suppose x is minimal. Then $\overline{\{x\}} = \{x\}$, because if not, then any $y \notin \overline{\{x\}} \setminus \{x\}$ is such that $x > y$; namely, $y \in \overline{\{x\}}$ but $x \notin \overline{\{y\}}$ (because $\overline{\{y\}}$ must be a strict subset of $\overline{\{x\}}$ and $\overline{\{x\}}$ is the smallest closed set containing x). Hence the result is shown in both directions.

(2) The maximal points are the generic points of the irreducible components of X.
Proof: Suppose that x is a generic point of an irreducible component V of X. Now suppose y is a point

contained with $y \geq x$. Then $x \in \overline{\{y\}}$, which implies $V = \overline{\{x\}} \subseteq \overline{\{y\}}$. Now y is contained in an irreducible component W; since W is a closed set containing y, $\overline{\{y\}} \subseteq W$. Since V is an irreducible component, this implies $V = W$, so $\overline{\{x\}} = \overline{\{y\}}$, so $x = y$ and x is maximal. On the other hand, suppose that x is a maximal point contained in an irreducible component V. Then there exists a generic point v of V, and since $x \in \overline{\{v\}}$, we have $x \leq v$ whence $x = v$ by maximality. Hence x is the generic point of an irreducible component of X.

(e): Let X be a Zariski space. Then,

(1) A closed subset contains every specialization of any of its points.
Proof: Suppose that V is a closed subset of X, and suppose that $v \in V$ is a point. Then a specialization of v is a point w such that $w \in \overline{\{v\}}$. But since $\overline{\{v\}}$ is the smallest closed set containing v, and V contains v, we must have $\overline{\{v\}} \subseteq V$. Hence $w \in \overline{\{v\}}$ implies $w \in V$, so V contains every specialization of v, as desired.

(2) An open subset contains every generization of any of its points.
Proof: Suppose that U is an open subset of X, and suppose that $u \in U$ is a point. Then a generization of u is a point v such that $u \in \overline{\{v\}}$. Suppose that $v \notin U$. Then $X \setminus U$ is a closed set containing v, so since $\overline{\{v\}}$ is the smallest closed set containing v, $\overline{\{v\}} \subseteq X \setminus U$. In summary, by contraposition, if $\overline{\{v\}}$ intersects U then $v \in U$. Yet $\overline{\{v\}}$ and U both contain u, so $v \in U$. Hence U contains every generization of u, as desired. □

Theorem 6.36. *If X is a scheme, every (nonempty) irreducible closed subset has a unique generic point.*

Suppose that Z is an irreducible closed subset of a scheme X. By the axioms of a scheme, we may select a nonempty affine open set $U \subseteq Z$. U is dense and irreducible in Z by Proposition 6.13 and Proposition 6.14. Now suppose $U = \operatorname{Spec} A$. It suffices to find a prime $\mathfrak{p} \lhd_{\text{pr}} A$ such that $\overline{\mathfrak{p}} = \operatorname{Spec} A = U$ in U, since then $\overline{(\mathfrak{p})} = \overline{U} = Z$ in Z. Hence let us attempt to find a prime $\mathfrak{p} \lhd_{\text{pr}} A$ such that $\overline{\mathfrak{p}} = \operatorname{Spec} A$. This is equivalent to finding a prime ideal $\mathfrak{p}$ such that any prime $\mathfrak{q} \lhd_{\text{pr}} A$ contains $\mathfrak{p}$. The nilradical is contained in every prime ideal, so it suffices to show that the nilradical is prime. Yet this is precisely Lemma 6.30.

We have shown the existence of a generic point, and will now show that this point is necessarily unique. Suppose that ζ_1 and ζ_2 are two generic points of an irreducible closed subset Z. Let U be an affine neighborhood of ζ_1. Since $\overline{\zeta_1} = \overline{\zeta_2}$, every open set containing ζ_1 contains ζ_2; hence U is also an affine neigborhood of ζ_2. By definition of affine, there exists an isomorphism of schemes $\phi : U \to \operatorname{Spec} A$ for some ring A. Then, $\zeta_2 \in \overline{\zeta_1}$ implies $\phi(\zeta_2) \in \overline{\phi(\zeta_1)}$ implies $\phi(\zeta_2) \in V(\phi(\zeta_1))$ implies $\phi(\zeta_2) \supseteq \phi(\zeta_1)$. The reverse reasoning implies that $\phi(\zeta_2) \subseteq \phi(\zeta_1)$, so $\phi(\zeta_1) = \phi(\zeta_2)$ whence $\zeta_1 = \zeta_2$, as desired.

Proposition 6.37. *If X is a Noetherian scheme, then $\operatorname{sp}(X)$ is a Noetherian topological space.*

Proof. Assume that X is a Noetherian scheme (Definition 2.74). This implies X is covered by a finite set of open affine subschemes $U_1, \ldots, U_n$ where $U_i = \operatorname{Spec} A_i$ for a Noetherian ring A_i. By Proposition 6.29, each U_i is a Noetherian space. Now, suppose we have an infinite ascending chain of open sets $V_1 \subseteq V_2 \subseteq V_3 \subseteq \cdots$. This gives us an infinite ascending chain of open sets $V_1 \cap U_i \subseteq V_2 \cap U_i \subseteq V_3 \cap U_i \subseteq \cdots$ within U_i; since U_i is Noetherian, there must exist N_i such that $V_{N_i} \cap U_i = V_{N_i+1} \cap U_i = \cdots$ for each i. But then define $N = \max\{N_1, \ldots, N_n\}$; here, we have $V_N = V_{N+1} = \cdots$ since the U_i cover X. Therefore the chain stabilizes. Hence $\operatorname{sp}(X)$ is indeed a Noetherian topological space, as desired. □

Corollary 6.37.1. *If X is a Noetherian scheme, then $\operatorname{sp}(X)$ is a Zariski space.*

B.6 Constructible Subsets and Chevalley's Theorem

Definition 6.38 (Constructible Subset)**.** Let X be a topological space. A *constructible subset* of X is a subset which belongs to the smallest family $\mathfrak{F}$ of subsets such that (1) every open subset is in $\mathfrak{F}$, (2) a finite intersection of elements of $\mathfrak{F}$ is in $\mathfrak{F}$, and (3) the complement of an element of $\mathfrak{F}$ is in $\mathfrak{F}$.

Definition 6.39 (Locally Closed)**.** A subset of X is *locally closed* if it is in the intersection of an open subset with a closed subset.

Theorem 6.40. *A subset of X is constructible if and only if it can be written as a finite union of locally closed subsets if and only if it can be written as a finite* disjoint *union of locally closed subsets.*

Proof. Let $\mathscr{F}$ be the family of constructible subsets, $\mathscr{G}$ be the family of subsets of X which can be written as a finite union of locally closed subsets, and $\mathscr{H}$ be the family of subsets of X which can be written as a finite *disjoint* union of locally closed subsets. We will show that $\mathscr{F} = \mathscr{G}$ and $\mathscr{G} = \mathscr{H}$.

(1) $\mathscr{F} = \mathscr{G}$**:** To show that $\mathfrak{G} \subseteq \mathfrak{F}$, first notice that the finite union of elements in $\mathfrak{F}$ is in $\mathfrak{F}$. To see why, suppose that $S, T \in \mathfrak{F}$. Then $X \setminus S, X \setminus T \in \mathfrak{F}$, whence $(X \setminus S) \cap (X \setminus T) \in \mathfrak{F}$, whence $X \setminus ((X \setminus S) \cap (X \setminus T)) = S \cup T \in \mathfrak{F}$. This immediately implies that $\mathfrak{G} \subseteq \mathfrak{F}$. To see why, notice that any open or closed set is in $\mathscr{F}$, so the intersection of an open subset with a closed subset is in $\mathscr{F}$, so the finite union of any locally closed subsets is constructible by our above reasoning that $\mathfrak{F}$ is closed under finite unions. Hence $\mathfrak{G} \subseteq \mathfrak{F}$.

Conversely, we will show that $\mathfrak{F} \subseteq \mathfrak{G}$. To do this, because $\mathfrak{F}$ is the smallest family of subsets which satisfy the properties (1)-(3), it suffices to show that $\mathfrak{G}$ satisfies (1)-(3). Now, plainly every open subset is in $\mathfrak{G}$, since any open set U can be written as the locally closed subset $U \cap X$. Similarly, the complement of an element of $\mathfrak{G}$ is in $\mathfrak{G}$. To see why, suppose that $(U_1 \cap V_1) \cup \cdots \cup (U_n \cap V_n)$ is a finite union of locally closed subsets (that is, the U_i are open and the V_i are closed). Then

$$X \setminus ((U_1 \cap V_1) \cup \cdots \cup (U_n \cap V_n)) = (X \setminus (U_1 \cap V_1)) \cap \cdots \cap (X \setminus (U_n \cap V_n)) = ((X \setminus U_1) \cup (X \setminus V_1)) \cap \cdots \cap ((X \setminus U_n) \cup (X \setminus V_n))$$

but notice that $X \setminus U_i$ is closed and $X \setminus V_i$ is open for each i, so by distributing and then grouping we can express $X \setminus ((U_1 \cap V_1) \cup \cdots \cup (U_n \cap V_n))$ as a finite union of locally closed subsets. Finally, obviously a finite union of elements of $\mathfrak{G}$ is in $\mathfrak{G}$; combined with the complement property, this implies that a finite intersection of elements of $\mathfrak{G}$ is in $\mathfrak{G}$. Hence $\mathfrak{G}$ satisfies (1)-(3), so $\mathfrak{F} \subseteq \mathfrak{G}$ and we are done.

(2) $\mathscr{G} = \mathscr{H}$**:** Now, obviously $\mathscr{H} \subseteq \mathscr{G}$; it suffices to argue the other direction: namely, that any finite union of locally closed subsets can be rewritten as a finite disjoint union of locally closed subsets. For this, it suffices to show that if $U \cap V$ and $U' \cap V'$ are two locally closed subsets (that is, U and U' are open, and V and V' are closed) with possibly nonempty intersection, then $(U \cap V) \cup (U' \cap V')$ can be written as a finite disjoint union of locally closed subsets. Yet in fact we have

$$\begin{aligned}(U \cap V) \cup (U' \cap V') =& (U \cap V \cap (X \setminus U') \cap (X \setminus V')) \cup (U \cap V \cap (X \setminus U') \cap V') \cup (U \cap V \cap U' \cap (X \setminus V')) \cup \\ & (U \cap V \cap U' \cap V') \cup (U \cap (X \setminus V) \cap U' \cap V') \cup ((X \setminus U) \cap V \cap U' \cap V') \cup \\ & ((X \setminus U) \cap (X \setminus V) \cap U' \cap V').\end{aligned}$$

and each of these subsets are plainly disjoint, and by grouping one may show that each of these are locally closed subsets. Hence $\mathscr{G} \subseteq \mathscr{H}$, so $\mathscr{G} = \mathscr{H}$. ☐

Proposition 6.41. *A constructible subset of an irreducible Zariski space is dense if and only if it contains the generic point. Furthermore, in that case it contains a nonempty open subset.*

Proof. Suppose that S is a constructible subset of an irreducible Zariski space X. Let x be the generic point of X. Now, if $x \in S$, then S is obviously dense as $\overline{S} \supseteq \overline{\{x\}} = X$. On the other hand, suppose that S is dense; that is, $\overline{S} = X$. Now, write S as a finite union of locally closed subsets $(U_1 \cap V_1) \cup \cdots \cup (U_n \cap V_n)$; we may assume that the U_i are nonempty for each i. Now, $V_1 \cup \cdots \cup V_n$ contains S and is closed, so it must contain $\overline{S} = X$. Hence $V_1 \cup \cdots \cup V_n = X$. By irreduciblity, this implies that $X = V_i$ for some i; without loss of generality, assume $X = V_1$. But then $S = U_1 \cup (U_2 \cap V_2) \cup \cdots \cup (U_n \cap V_n)$. Hence S contains a nonempty open set U_1, which by Theorem 6.35(c) contains the generic point x of X. Hence S contains a nonempty open set U_1 and the generic point x, completing both parts of the question. ☐

Proposition 6.42. *Suppose that X is a Zariski space. A subset S of X is closed if and only if it is constructible and stable under specialization. Similarly, a subset T of X if and only if it is constructible and stable under generization.*

Proof. Firstly, notice that a closed subset S of X is plainly constructible using axiom (1) and (3), and recall that it is stable under specialization by Theorem 6.35(e). Hence assume that S is a constructible subset which is stable under specialization. Then, we can write S as a finite union of locally closed subsets

$(U_1 \cap V_1) \cup \cdots \cup (U_n \cap V_n)$ (as usual, the U_i are open and the V_i are closed).

Now, as a Zariski space, X is Noetherian, so any subspace of X is also Noetherian by Proposition 6.21. Recall that any Noetherian space has finitely many irreducible components. Therefore, by splitting each V_i up into irreducible components C_{ij}, and ignoring the ones that have empty intersection with U_i, we may assume $S = (U_1 \cap V_1') \cup \cdots \cup (U_n \cap V_n')$ for open U_i and irreducible closed V_i' such that $U_i \cap V_i'$ is nonempty for each i (key here is the fact that an irreducible component of V_i is also closed and irreducible in X).

Now, I claim that in fact $S = V_1' \cup \cdots \cup V_n'$ and hence that S is closed. To see why this holds, notice that clearly $S \subseteq V_1' \cup \cdots \cup V_n'$. On the other hand, notice that S contains $U_i \cap V_i'$, which is a nonempty open subset of V_i', and therefore contains the generic point v_i of V_i' by II.3.17(d). But then since S is closed under specialization, S contains every point in $\overline{\{v_i\}} = V_i'$. Hence S contains V_i' for each i, so S contains $V_1' \cup \cdots \cup V_n'$ and we have $S = V_1' \cup \cdots \cup V_n'$, as desired.

For the second part of this result, notice that an open subset S is plainly constructible using axiom (1), and recall that it is stable under generization by Theorem 6.35(e). For the converse, assume that S is a constructible subset which is stable under generization. Then $X \setminus S$ is a constructible subset which is stable under specialization, so it is closed by the above reasoning, so S is open. □

Proposition 6.43. *If $f : X \to Y$ is a continuous map of Zariski spaces, then the inverse image of any constructible subset of Y is a constructible subset of X.*

Proof. Suppose that $(U_1 \cap V_1) \cup \cdots \cup (U_n \cap V_n)$ is a finite disjoint union of locally closed subsets. Then

$$\begin{aligned} f^{-1}\left((U_1 \cap V_1) \cup \cdots \cup (U_n \cap V_n)\right) &= f^{-1}(U_1 \cap V_1) \cup \cdots \cup f^{-1}(U_n \cap V_n) \\ &= (f^{-1}(U_1) \cap f^{-1}(V_1)) \cup \cdots \cup (f^{-1}(U_n) \cap f^{-1}(V_n)). \end{aligned}$$

But $f^{-1}(U_i)$ is open and $f^{-1}(V_i)$ is closed by continuity, so $(f^{-1}(U_1) \cap f^{-1}(V_1)) \cup \cdots \cup (f^{-1}(U_n) \cap f^{-1}(V_n))$ is indeed constructible. Hence we are done. □

We have been building up all this machinery for the proof of the following theorem:

Theorem 6.44 (Chevalley's Theorem)**.** *Let Y be a Noetherian scheme and $f : X \to Y$ be a morphism of finite type. Then the image of any constructible subset of X is a constructible subset of Y. In particular, $f(X)$, which need not be either open or closed, is a constructible subset of Y.*

Firstly, we need a special tool: Noetherian induction. We will state the result, but will not include a proof as it is simple (it is a special case of the general method of induction on well-founded sets).

Theorem 6.45 (Noetherian Induction)**.** *Let X be a Noetherian topological space and let $\mathscr{P}$ be a property of closed subsets of X. Suppose that if $Y \subseteq X$ is closed, then if $\mathscr{P}$ holds for every proper closed subset of Y, then $\mathscr{P}$ holds for Y. In particular, $\mathscr{P}$ must hold for the empty set by vacuous truth. Then $\mathscr{P}$ holds for X.*

Secondly, we need an algebraic lemma from Atiyah-Macdonald:

Lemma 6.46. *Let $A \subseteq B$ be an inclusion of Noetherian integral domains such that B is a finitely-generated A-algebra. Then, given any nonzero $b \in B$, there is a nonzero element $a \in A$ such that the following property holds: if $\varphi : A \to K$ is any homomorphism of A to an algebraically closed field K such that $\varphi(a) \neq 0$, then φ extends to a homomorphism φ' of B into K such that $\varphi'(b) \neq 0$.*

Proof. Let g be the number of generators of B over A; we prove this result by induction on g. Consider the base case of $g = 1$, so that there exists $x \in B$ such that $A[x] = B$. In particular, any nonzero $b \in B$ can be expressed as $a_n x^n + \cdots + a_1 x + a_0$ for some $n \geq 0$ and some $a_n, \ldots, a_0 \in A$. Now, there are two possibilities:

1. x is transcendental over A. In this case, I claim that any homomorphism $\varphi : A \to K$ such that $\varphi(a_n) \neq 0$ can be extended to a homomorphism $\varphi' : B \to K$ with $\varphi'(b) \neq 0$. It suffices to choose $\varphi'(x) = y$ such that $\varphi(a_n)y^n + \cdots + \varphi(a_1)y + \varphi(a_0) \neq 0$. Yet since $\varphi(a_n) \neq 0$, $\varphi(a_n)y^n + \cdots + \varphi(a_1)y + \varphi(a_0)$ is a degree n polynomial, so it has at most n roots. Since K is algebraically closed, it is infinite, so we may choose any of the infinite not-roots of $a_n y^n + \cdots + a_1 y + a_0$ to be y. The result follows.

2. x is algebraic over A. Then x satisfies $a'_m x^m + a'_{m-1}x^{m-1} + \cdots + a'_1 x + a'_0 = 0$ for some m and $a'_i \in A$. Similarly, since x is algebraic over A, $\text{Frac}(B)$ is algebraic over A, so in particular b^{-1} is algebraic over A. Hence $a''_k(b^{-1})^k + a''_{k-1}(b^{-1})^{k-1} + \cdots + a''_1(b^{-1}) + a''_0 = 0$ for some k and $a''_i \in A$.

   Now let $a = a'_m a''_k$. Choose any homomorphism $f : A \to \Omega$ such that $f(a) \neq 0$. Then φ can be extended to a map $A_a \to \Omega$ given by $a^{-1} \mapsto f(a)^{-1}$ and then to a map $R \to \Omega$, where R is a valuation ring containing A_a (via the general result on extending homomorphisms to valuation rings). Then x is integral over A_a (because a'_m is invertible in A_a), so x is in R, so $B \subseteq R$. Therefore we may restrict the map $R \to \Omega$ to a map $\varphi' : B \to \Omega$. Now b^{-1} is integral over A_a (because a''_k is invertible in A_a), so b^{-1} is in R. Hence b is a unit in R, which forces $\varphi'(b) = \varphi(b) \neq 0$. The result follows.

Now, the result follows easily by induction; we progressively extend the homomorphism to an A-subalgebra of B which uses one more generator each time until we have any finite number of generators. ☐

Now we can begin the proof of Chevalley's Theorem:

Proof. First, there are five reductions to be made:

1. To reduce to showing that $f(X)$ itself is constructible, restrict to the morphism $f|_S : S \to Y$.

2. To reduce to the case where X and Y are affine, suppose that $\{U_i\}$ is a (finite, by Noetherianness) affine open cover of Y. Then suppose that, for each i, $\{V_{ij}\}$ is a (finite, since f is of finite type) affine open cover of $f^{-1}(U_i)$. If the morphism $f|_{V_{ij}} : V_{ij} \to U_i$ has $f(V_{ij})$ constructible for all i, j, then $f(X) = \bigcup_{i,j} f(V_{ij})$ is constructible as it is a finite union of constructible sets. Yet $f|_{V_{ij}}$ maps into the affine scheme U_i, so we may assume that both X and Y are affine.

3. Similarly, suppose that $\{V_i\}$ are the irreducible components of Y. Then, suppose that, for each i, $\{W_{ij}\}$ are the irreducible components of $f^{-1}(V_i)$. If the morphism $f|_{W_{ij}} : W_{ij} \to V_i$ has $f(W_{ij})$ constructible for all i, j, then $f(X) = \bigcup_{i,j} f(W_{ij})$ is constructible. Yet $f|_{W_{ij}}$ maps into the irreducible component V_i, so we may assume that both X and Y are irreducible. Since X and X_{red} are homeomorphic for all schemes X, we can plainly take X and Y to be reduced. Hence we may assume that X and Y are irreducible and reduced; that is, that they are integral.

4. X is Noetherian (we are not even reducing cases, but stating a consequence of our above work), as the morphism of finite type $\text{Spec}(X) \to \text{Spec}(Y)$ gives a finite ring homomorphism $Y \to X$, and since Y is Noetherian, X is Noetherian as a finitely-generated algebra over a Noetherian ring.

5. Now, we will reduce to the case where $f : X \to Y$ is dominant. To do this, assume that we have shown the result for every dominant morphism. Now, for any morphism $f : X \to Y$ where X and Y are affine integral notherian schemes, we have an induced morphism $f' : X \to \overline{f(X)}$. f' is dominant, so $f'(X)$ is constructible by assumption. By Theorem 6.40, $f'(X)$ can be expressed as a finite disjoint union of locally closed subsets $(U_1 \cap V_1) \cup \cdots \cup (U_n \cap V_n)$. Note that the U_i are open and the V_i are closed in $\overline{f(X)}$, but since $\overline{f(X)}$ is closed, the V_i are still closed in X, and the U_i can be expressed as $U'_i \cap \overline{f(X)}$ for some U'_i open in X. Hence $f'(X) = f(X) = (U_1 \cap (V_1 \cap \overline{f}(X))) \cup \cdots \cup (U_n \cap (V_n \cap \overline{f}(X)))$, so indeed $f(X)$ is constructible. This shows that it suffices to prove the result for dominant f.

Therefore, we have reduced to showing that $f(X)$ itself is constructible, in the case where $X = \text{Spec}\, B$ and $Y = \text{Spec}\, A$ are affine, integral noetherian schemes, and f is a dominant morphism.

Next, we will show that $f(X)$ contains a nonempty open subset of Y. The morphism $f : X \to Y$ of finite type corresponds to a morphism $f' : A \to B$ which is an injection (since f is dominant) making B into a finitely-generated A-algebra. By our reductions in (a), A and B are Noetherian integral domains. Now, to show that $f(X)$ contains a nonempty open subset of Y, choose $b = 1$. Then there exists some nonzero $a \in A$ with the property described in Lemma 6.46.

It suffices to show that $f(X)$ contains $D(a)$, since $D(a)$ is nonempty (since $a \neq 0$) and open. Now suppose that $\mathfrak{p} \in D(a)$; that is, $\mathfrak{p}$ does not contain a. Then the natural map $\varphi : A \to A/\mathfrak{p} \hookrightarrow \text{Frac}(A/\mathfrak{p}) \hookrightarrow \overline{\text{Frac}(A/\mathfrak{p})}$

sends a to a nonzero element of the algebraically closed field $\overline{\mathrm{Frac}(A/\mathfrak{p})}$. Hence we have an extended map $\varphi : B \to \overline{\mathrm{Frac}(A/\mathfrak{p})}$ sending 1 to itself. Then $\ker \varphi$ is a prime ideal of φ (notably it is not all of B since $\varphi(1)$ is nonzero) which contains $\mathfrak{p}$, so $\mathfrak{p} = f'^{-1}(\ker \varphi)$ whence $f(\ker \varphi) = \mathfrak{p}$ so $\mathfrak{p} \in f(X)$, as desired.

Finally, we can conclude the result using Noetherian induction. Say that $\mathscr{P}$ holds for a closed subset Y of X if $f(Y)$ is constructible. Plainly, $\mathscr{P}$ holds for the empty set. Now assume that $\mathscr{P}$ holds for every proper closed subset of Y; it suffices to prove that $\mathscr{P}$ holds for Y. Now, obviously if Y is reducible, then Y can be written as the union of two constructible sets, so we are done. Therefore, we may assume that Y is irreducible. Now, consider $f|_Y : Y \to f(Y)$; by (b), there is a nonempty open set $U \subseteq f(Y)$. Now take $P = (f|_Y)^{-1}(U) \subseteq Y$; this is open by continuity (and obviously nonempty). Hence $Y \setminus P$ is a proper closed subset of Y, so $f(Y \setminus P)$ is constructible, but $f(Y) = f(Y \setminus P) \cup U$ so $f(Y)$ is also constructible, as desired. Therefore we are done, and Chevalley's Theorem is proven. $\square$