An Introduction to Category Theory
(and a little bit of algebraic topology)

Robin Truax

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1 What is Category Theory?

> "[Category theory] does not itself solve hard problems in topology or algebra. It clears away tangled multitudes of individually trivial problems. It puts the hard problems in clear relief and makes their solution possible." – Colin McLarty

A *category*, in its full generality, is not much more than a generalization of a labeled directed multigraph – a class of *objects* and a class of *arrows* (also known as *morphisms*) between them. Category theory is used in a variety of subfields of math both to unify certain "natural" definitions and for the tools it can help develop. Its origins lie in mid-20th algebraic topology, but the subject has exploded in the less than a century since its development.

In particular, they are used in homological and cohomological algebra, algebraic geometry, and algebraic topology. They appear as knot invariants, abstract vector spaces, and more. They are also used outside of pure math: in mathematical physics, biology, and (especially) computer science, where categories provide a useful language for functional programming.

Sándor has often said that the most important distinction between different fields of math are the types of functions that you consider – linear transformations, homeomorphisms, automorphisms, and so on. I don't think I quite understood what he meant until I began to learn category theory. The maxim to remember is that "an object is determined by its relationships to other objects" – succinctly, "it's all about the maps".

2 Basic Definitions

Definition 2.1 (Categories)**.** A *category* $\mathbf{C}$ is a class of *objects* $\mathrm{Ob}(\mathbf{C})$ and *morphisms* or *arrows* $\mathrm{Mor}(\mathbf{C})$ satisfying the following requirements:

1. For each morphism f, there are objects $\mathrm{dom}(f) = A$ and $\mathrm{cod}(f) = B$, called the *domain* and *codomain* of f. In this case, we write $f : A \to B$
2. Given any two morphisms $f : A \to B$ and $g : B \to C$, there exists an morphism $g \circ f : A \to C$, called the *composition* of f and g.
3. Given any object A, there is an *identity morphism* $1_A : A \to A$ such that for any $f : A \to B$, $f \circ 1_A = f = 1_B \circ f$.
4. Morphism composition is associative: given any three morphisms $f : A \to B$, $g : B \to C$, $h : C \to D$, $(f \circ g) \circ h = f \circ (g \circ h)$.

Definition 2.2 (Small and Locally Small)**.** A category $\mathbf{C}$ is called *small* if both $\mathrm{Ob}(\mathbf{C})$ and $\mathrm{Mor}(\mathbf{C})$ are sets. Otherwise, $\mathbf{C}$ is called *large*. Similarly, a category $\mathbf{C}$ is called *locally small* if for all objects $X, Y \in \mathbf{C}$, the collection $\mathrm{Hom}_{\mathbf{C}}(X, Y)$ of morphisms $X \to Y$ is a set (called a *hom-set*).

Definition 2.3 (Isomorphism of Objects)**.** In any category $\mathbf{C}$, an morphism $f : A \to B$ is called an *isomorphism* if there is a morphism $g : B \to A$ such that $f \circ g = 1_B$ and $g \circ f = 1_A$. In this case, f and g are called *inverses*, g is denoted f^{-1}, and we say A is *isomorphic* to B (denoted $A \cong B$).

Following are some important examples of categories:

- **Sets** is the category of sets and functions between them.
- $\mathbf{Sets}_{\mathit{fin}}$ is the category of finite sets and functions between them.
- **Groups** is the category of groups and group homomorphisms.
- **Rings** is the category of rings (with unity) and ring homomorphisms (which preserve 1)

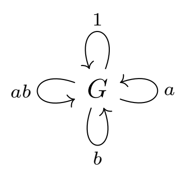
- **Graphs** is the category of graphs and graph homomorphisms.
- $\mathbf{Vect_k}$ is the category of vector spaces over a field k and k-linear transformations.
- $\mathbf{Mod_R}$ is the category of modules over a ring R and R-module homomorphisms.
- **Top** is the category of topological spaces and continuous mappings.

One of the most important examples of a category is a *poset* (a partially ordered set). Here, the objects of the poset category are simply the elements of the set, and the arrows $f : x \to y$ correspond to orderings $x \leq y$. For example, $\mathbf{Ord}_{fin}$, the set of finite ordinals (specifically the von Neumann ordinals) is naturally a category by the usual ordering.

It will also be helpful for us to define the following categories: **0** is the empty category (with no objects and no morphisms), **1** is the category with one object and the identity morphism, and **2** is the following category:

$$\mathrm{id}_A \circlearrowright A \xrightarrow{f} B \circlearrowleft \mathrm{id}_B$$

Finally, an individual group is itself a category with exactly one object, where all the morphisms are isomorphisms. For a given group G, this category is called $\mathbf{B}G$. For example, the category $\mathbf{B}\{1\}$ (where $\{1\}$ is the trivial group) is **1** and the category $\mathbf{B}V_4$ (where V_4 is the Klein four-group) is:



Of course, you can figure out how the composition is defined.

Definition 2.4 (Opposite Category)**.** The *opposite category* $\mathbf{C}^{\mathrm{op}}$ of a category $\mathbf{C}$ is formed by simply "reversing all the arrows".

Definition 2.5 (Discrete Category)**.** Let X be a class of objects. The *discrete category* $\mathbf{Dis(X)}$ is the category formed using X for the class of objects and only adding the required identity morphisms for each object $O \in X$.

Definition 2.6 (Arrow Category)**.** The *arrow category* or *morphism category* $\mathcal{C}^{\to}$ of a category $\mathcal{C}$ has the morphisms of $\mathcal{C}$ as objects, and a morphism g from $f : A \to B$ to $f' : A' \to B'$ is a pair of morphisms (g_1, g_2) in $\mathcal{C}$ such that $g_2 \circ f = f' \circ g_1$ – i.e. such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{g_1} & A' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ B & \xrightarrow{g_2} & B' \end{array}$$

Definition 2.7 (Subcategory)**.** A *subcategory* of a category $\mathbf{C}$ is a subclass $\mathrm{Ob}(\mathbf{D}) \subseteq \mathrm{Ob}(\mathbf{C})$ and a subclass $\mathrm{Mor}(\mathbf{D}) \subseteq \mathrm{Mor}(\mathbf{C})$ such that any morphism in $\mathrm{Mor}(\mathbf{D})$ is between two objects in $\mathrm{Ob}(\mathbf{D})$. For example, **0** is a subcategory of any category.

2.1 Types of Morphisms

Definition 2.8 (Monomorphisms)**.** A morphism $f : A \to B$ is called a *monomorphism* if it is left-cancellative – i.e. $fg = fh \Rightarrow g = h$. In this case, we write $f : A \rightarrowtail B$ and say f is monic.

Definition 2.9 (Epimorphisms)**.** An *epimorphism* if it is right-cancellative – i.e. $gf = hf \Rightarrow g = h$. In this case, we write $f : A \twoheadrightarrow B$ and say f is epic.

Definition 2.10 (Bimorphism)**.** A morphism if called a *bimorphism* if it is both epic and monic.

Definition 2.11 (Retractions and Sections)**.** A morphism is called a *retraction* if it has a left-inverse and a *section* if it has a right-inverse. Note that a morphism which is both a retraction and a section is an isomorphism.

Notice that in **Sets**, the monomorphisms are exactly the retractions (i.e. the injective functions) and the epimorphisms are exactly the surjections (i.e. the surjective functions). Furthemore, the bimorphisms correspond exactly to the isomorphisms (i.e. the bijections). However, this is not generally the case: being an isomorphism is a strictly stronger condition than being a bimorphism. Still, it inspires the following definition:

Definition 2.12 (Subobject)**.** A *subobject* of an object X in $\mathcal{C}$ is a monomorphism $m : M \rightarrowtail X$.

Proposition 1. *All isomorphisms are bimorphisms (that is, they are monic and epic). More generally, any retraction is monic and any section is epic. In contrast, not all bimorphisms are isomorphisms.*

Proof. The first two parts of the proposition are trivial. To see an example of a bimorphism which is not an isomorphism, consider the example of the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ in the category **Rings**. This is both a monomorphism and epimorphism, but not an isomorphism. □

Definition 2.13 (Endomorphisms and Automorphisms)**.** An *endomorphism* is a morphism $f : A \to A$ from an object to itself. If an endomorphism is also an isomorphism, then it is called an *automorphism*. The class of endomorphisms of an object A is denoted $\mathrm{End}(A)$ and the class of automorphisms is denoted $\mathrm{Aut}(A)$.

Definition 2.14 (Terminal, Initial, and Zero Objects)**.** In a category $\mathbf{C}$, an object T is *terminal* if for any object $C \in \mathbf{C}$, there is a unique morphism $C \to T$. Similarly, an object I is *initial* if for any object $C \in \mathbf{C}$, there is a unique morphism $T \to C$. An object which is both terminal and initial is called a *zero object*.

Following are some examples of terminal, initial, and zero objects:

- The empty set is initial in the category **Sets**. The terminal objects in **Sets** are the singleton sets – the sets with just one element.
- The trivial group is a zero object in **Groups**.
- $\mathbb{Z}$ is initial in the category **Rings**, and the zero ring (which, confusingly, is not a zero object) is the terminal object.

Proposition 2. *Terminal objects in a category* $\mathbf{C}$ *(if they exist) are unique up to unique isomorphism. Similarly, if they exist, initial and zero objects are unique up to unique isomorphism.*

2.2 A Functor

Definition 2.15 (Covariant Functor)**.** A *covariant functor* (or just a *functor*) $\mathcal{F} : \mathbf{C} \to \mathbf{D}$ between categories $\mathbf{C}$ and $\mathbf{D}$ is a mapping $\mathrm{Ob}\,\mathbf{C} \to \mathrm{Ob}\,\mathbf{D}$ and $\mathrm{Mor}\,\mathbf{C} \to \mathrm{Mor}\,\mathbf{D}$ such that:

- $\mathcal{F}(f : A \to B) = \mathcal{F}(f) : \mathcal{F}(A) \to \mathcal{F}(B)$,
- $\mathcal{F}(1_A) = 1_{\mathcal{F}(A)}$, and
- $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$.

In short, a functor is a morphism of categories. In particular, every category has the identity functor $1_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$. Thus, we can define **Cat** as the (locally small) category of small categories and functors between them. This lets us define isomorphic categories: categories that are isomorphic is the category **Cat**. More formally:

Definition 2.16 (Isomorphism of Categories)**.** Two categories $\mathbf{C}$ and $\mathbf{D}$ are *isomorphic* if there exist functors $\mathcal{F} : \mathbf{C} \to \mathbf{D}$ and $\mathcal{G} : \mathbf{D} \to \mathbf{C}$ that are inverses ($\mathcal{F} \circ \mathcal{G} = 1_{\mathbf{D}}$ and $\mathcal{G} \circ \mathcal{F} = 1_{\mathbf{C}}$). In particular, a functor is an isomorphism functor if and only if it is bijective on the class of objects and the class of morphisms.

Definition 2.17 (Properties of Functors)**.** The following are properties that a functor $\mathscr{F} : \mathbf{C} \to \mathbf{D}$ can have:

- $\mathscr{F}$ is *injective on objects* if the object map $\mathscr{F}_0 : \operatorname{Ob}\mathbf{C} \to \operatorname{Ob}\mathbf{D}$ is injective.
- $\mathscr{F}$ is *surjective on objects* if the object map $\mathscr{F}_0 : \operatorname{Ob}\mathbf{C} \to \operatorname{Ob}\mathbf{D}$ is surjective.
- $\mathscr{F}$ is *injective on morphisms* if the morphism map $\mathscr{F}_0 : \operatorname{Mor}\mathbf{C} \to \operatorname{Mor}\mathbf{D}$ is injective.
- $\mathscr{F}$ is *surjective on morphisms* if the morphism map $\mathscr{F}_0 : \operatorname{Mor}\mathbf{C} \to \operatorname{Mor}\mathbf{D}$ is surjective.
- $\mathscr{F}$ is *faithful* if for all $A, B \in \operatorname{Ob}\mathbf{C}$, the map $F_{A,B} : \operatorname{Hom}_{\mathbf{C}}(A, B) \to \operatorname{Hom}_{\mathbf{D}}(\mathscr{F}(A), \mathscr{F}(B))$ is injective.
- $\mathscr{F}$ is *full* if $F_{A,B}$ is always surjective.

The idea of a faithful functor is particularly interesting, as it allows us to formalize our notion that many categories are simply categories of "sets with extra structure".

Definition 2.18 (Concrete Categories)**.** A *concrete category* is a pair $(\mathbf{C}, \mathcal{F})$ where $\mathbf{C}$ is a category and $\mathcal{F} : \mathbf{C} \to \mathbf{Sets}$ is a faithful functor. In particular, $(\mathbf{Groups}, \mathcal{F}_{\mathbf{Groups}}), (\mathbf{Rings}, \mathcal{F}_{\mathbf{Rings}}), (\mathbf{Top}, \mathcal{F}_{\mathbf{Top}})$ are all concrete categories with the usual forgetful functors.

Definition 2.19 (Subcategories)**.** We can generalize our earlier notion of a subcategory to match with our definition of subobject: a *subcategory* of a category $\mathbf{C}$ is a category $\mathbf{C}'$ such that there is an injective functor from $\mathbf{C}' \hookrightarrow \mathbf{C}$.

Definition 2.20 (Full Subcategories)**.** A *full subcategory* of a category $\mathbf{C}$ is a subclass of objects in $\mathbf{C}$ and *all* the arrows between them. For example, the category $\mathbf{Sets}_{\mathit{fin}}$ is a full subcategory of $\mathbf{Sets}$ but $\mathbf{Groups}$ is not a full subcategory of $\mathbf{Sets}$.

Definition 2.21 (Contravariant Functor)**.** A *contravariant functor* $\mathcal{F} : \mathbf{C} \to \mathbf{D}$ is exactly the same, except we "reverse the arrows". Formally, a covariant functor is a mapping $\operatorname{Ob}(\mathbf{C}) \to \operatorname{Ob}(\mathbf{D})$ and $\operatorname{Mor}(\mathbf{C})$ and $\operatorname{Mor}(\mathbf{D})$ such that:

- $\mathcal{F}(f : A \to B) = \mathcal{F}(f) : \mathcal{F}(B) \to \mathcal{F}(A)$,
- $\mathcal{F}(1_A) = 1_{\mathcal{F}(A)}$, and
- $\mathcal{F}(g \circ f) = \mathcal{F}(f) \circ \mathcal{F}(g)$ for all composable $f, g \in \operatorname{Mor}(\mathbf{C})$.

Lemma 3. *For any contravariant functor $F : \mathbf{C} \to \mathbf{D}$, there are covariant functors $F' : \mathbf{C}^{op} \to \mathbf{D}$ and $F'' : \mathbf{C} \to \mathbf{D}^{op}$.*

2.3 A Natural Transformation

Definition 2.22 (Natural Transformation)**.** A *natural transformation* is a "morphism of functors". More formally, for categories $\mathbf{C}$ and $\mathbf{D}$ with morphisms $\mathcal{F}, \mathcal{G} : \mathbf{C} \to \mathbf{D}$, a natural transformation $\vartheta : \mathcal{F} \to \mathcal{G}$ is a family of morphisms in $\mathbf{D}$, $(\vartheta_C : \mathcal{F}(C) \to \mathcal{G}(C))_{C \in \operatorname{Ob}(\mathbf{C})}$, such that, for any $f : C \to C'$ in $\mathbf{C}$, the following diagram commutes:

$$\begin{array}{ccc} \mathscr{F}(C) & \xrightarrow{\vartheta_C} & \mathscr{G}(C) \\ {\scriptstyle \mathscr{F}(f)}\big\downarrow & & \big\downarrow{\scriptstyle \mathscr{G}(f)} \\ \mathscr{F}(C') & \xrightarrow[\vartheta_{C'}]{} & \mathscr{G}(C') \end{array}$$

Then the $\mathcal{D}$-arrow $\vartheta_C : \mathscr{F}(C) \to \mathscr{G}(C)$ is called the *component of* ϑ *at* C.

Definition 2.23 (The Functor Category)**.** The *functor category* $\mathbf{Fun}(\mathbf{C}, \mathbf{D})$ has functors $\mathcal{F} : \mathbf{C} \to \mathbf{D}$ as objects and natural transformations $\vartheta : \mathcal{F} \to \mathcal{G}$ as morphisms. In particular, a natural transformation is called a *natural isomorphism* if it is an isomorphism in this category (equivalently, if each component ϑ_C is an isomorphism).

Definition 2.24 (Equivalence of Categories)**.** An *equivalence of categories* is a pair of functors:

$$\mathscr{E} : \mathcal{C} \to \mathcal{D} \text{ and } \mathscr{F} : \mathcal{D} \to \mathcal{C}$$

and a pair of natural isomorphisms:

$$\alpha : 1_{\mathcal{C}} \xrightarrow{\sim} F \circ E \text{ and } \beta : 1_{\mathcal{D}} \xrightarrow{\sim} E \circ F$$

In this case, we write $\mathcal{C} \simeq \mathcal{D}$ (isomorphism of categories is written $\mathcal{C} \cong \mathcal{D}$).

This is not as strong as "isomorphism of categories", but it is actually more important: identities are much less important than isomorphisms (just like in group theory, for example), so we seek to push our equivalences to their limits.

For example, $\mathbf{Sets}_{fin} \simeq \mathbf{Ord}_{fin}$, but clearly they are not isomorphic as categories (in particular, $\mathbf{Sets}_{fin}$ is large whereas $\mathbf{Ord}_{fin}$ is not).

Proposition 4. *The following conditions on a functor* $\mathscr{F} : \mathcal{C} \to \mathcal{D}$ *are equivalent:*

1. $\mathscr{F}$ *is part of an equivalence of categories.*
2. $\mathscr{F}$ *is full and faithful and "essentially surjective" on objects.*

Definition 2.25 (Essentially Small)**.** A category $\mathbf{C}$ is essentially small if it is equivalent to a small category – i.e. there is a set of objects so that every object in your category is isomorphic to one of these specific objects. For example, given our above result that $\mathbf{Sets}_{fin} \simeq \mathbf{Ord}_{fin}$, $\mathbf{Sets}_{fin}$ is essentially small. Similarly, the category of finitely-generated abelian groups is no small, but it is essentially small.

3 Limits and Colimits

At the core of defining constructions in categories is the idea of the "universal mapping property" or UMP – an idea which states that the final result of a particular construction, if it is natural, should satisfy a relationship with every other possible candidate result. In this section, we will create definitions and explore examples that will make this idea concrete.

3.1 Duality

Definition 3.1 (Dual Statements)**.** For any Σ, the *dual statement* Σ^* is formed by replacing $f \circ g$ with $g \circ f$, cod with dom, and dom with cod (reversing the direction and order of the composition of all morphisms).

But notice that the axioms for category theory are self-dual! That is, if we let CT denote the axioms of category theory, $\mathrm{CT}^* = \mathrm{CT}$. Thus, we get the following result about duality:

Proposition 5 (Duality)**.** *Formally, for any sentence* Σ *in the langauge of category theory, if* Σ *follows from the axioms of category theory, then so does its dual* Σ^**. More conceptually, if* Σ *is a true statement about categories, then so is* Σ^**.*

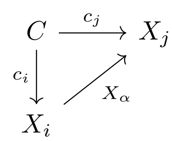
We usually use the prefix "co-" to denote the dual notion of an object or construction in a category. In the rest of this section (and indeed the rest of these notes), we will be exploring the co-notions of any notion that we come across.

3.2 Cones and Cocones

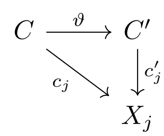
Definition 3.2 (Diagrams of Shape)**.** Let **J** be an "index category" (which has no special properties, like an index set, but is distinguished for convenience) and **C** be an arbitrary category. Then a *diagram of shape* (or *diagram of type*) **J** in **C** is a functor $\mathcal{X} : \mathbf{J} \to \mathbf{C}$.

We write the objects in the index category **J** using lowercase $i, j, \ldots$ and the value $\mathcal{X}(i) \in \mathbf{C}$ as X_i.

Definition 3.3. A *cone* to a diagram $\mathcal{X}$ consists of an object C in **C** and a family of arrows in **C**, $c_j : C \to X_j$ (for each $j \in \mathbf{J}$), such that for every arrow $\alpha : i \to j$ in **J**, the following triangle commutes:

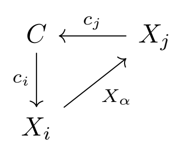


In particular, a *morphism of cones* $\vartheta : (C, c_j) \to (C', c'_j)$ is an arrow ϑ in **C** making the following triangle commute:

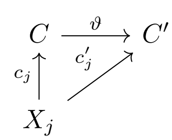


Thus, we can construct a category $\mathbf{Cone}(\mathcal{X})$.

Definition 3.4 (Cocone)**.** A *cocone* from the base $\mathcal{X}$ consists of an object C and a family of arrows in **C**, $c_j : X_j \to C$ for each $j \in \mathbf{J}$, such that for all $\alpha : i \to j \in \mathbf{J}$, the following triangle commutes:



In particular, a *morphism of cocones*: $\vartheta : (C, c_j) \to (C', c'_j)$ is an arrow ϑ in **C** making the following triangle commute:



Thus we can construct a category $\mathbf{Cocone}(\mathcal{X})$.

There is, actually, a much more "elegant" way to interpret these two concepts that avoids the use of universal properties, bundling it in the idea of natural transformations:

Definition 3.5 (Constant Functors)**.** Suppose we have a category **C** and an index category **J**. Then, for any object $A \in \mathbf{A}$, let $\mathcal{F}_A$ denote the constant functor $\mathbf{J} \to \mathbf{C}$ given by sending every element to A and every morphism to 1_A.

Proposition 6. *Given a diagram of shape* $\mathcal{X} : \mathbf{J} \to \mathbf{C}$*, a cone to* A *is given by a natural transformation* $\mathcal{F}_A \to \mathcal{X}$ *(and a cocone on* A *is given by a natural transformation* $\mathcal{F}_A \to \mathcal{X}$*).*

3.3 Limits and Colimits

Definition 3.6 (Limits)**.** A *limit* for a diagram $\mathcal{X} : \mathbf{J} \to \mathbf{C}$ is a terminal object in $\mathbf{Cone}(\mathcal{X})$ – there is a unique morphism from any other cone over $\mathcal{X}$ onto it. A *finite limit* is a limit for a diagram on a finite index category $\mathbf{J}$. In particular, the limiting cone can be thought of as the "closest cone over" the diagram D and is denoted $\varprojlim_{j \in \mathbf{J}} X_j$.

Definition 3.7 (Colimits)**.** A *colimit* for a diagram $\mathcal{X} : \mathbf{J} \to \mathbf{C}$ is an initial object in $\mathbf{Cocone}(\mathcal{X})$ – there is a unique morphism from it to any other cocone over $\mathcal{X}$. A *finite colimit* is a colimit for a diagram on a finite index category $\mathbf{J}$. In particular, the colimiting cone can be thought of as the "closest cone under" the diagram D and is denoted $\varinjlim_{j \in \mathbf{J}} X_j$.

Proposition 7 (Uniqueness of Limits and Colimits)**.** *A limit or colimit for a diagram* $\mathcal{X} : \mathbf{J} \to \mathbf{C}$*, if it exists, is unique up to unique isomorphism.*

Proof. This immediately follows from the proposition that terminal and initial objects are unique up to unique isomorphism. □

In particular, the terminal and initial object of a category $\mathbf{C}$ (if they exist) are the limits and colimits, respectively, of the empty diagram.

3.4 Direct and Inverse Limits

Perhaps the most intuitive use of limits is in the definition of direct and inverse limits.

Definition 3.8 (ω and ω^{op} as Index Categories)**.** Recall that $\omega = \langle \mathbb{N}, \leq \rangle$ is a poset – thus we may consider it as the category:

$$0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots$$

Also notice that ω^{op}, the poset $\langle \mathbb{N}, \geq \rangle$, is the category:

$$0 \longleftarrow 1 \longleftarrow 2 \longleftarrow 3 \longleftarrow \cdots$$

In particular, these are the index categories we use to define "direct and inverse limits".

Definition 3.9 (Direct Limits)**.** The *direct limit* (or *inductive limit*) of a sequence of objects and morphisms

$$A_0 \underset{a_0}{\longrightarrow} A_1 \underset{a_1}{\longrightarrow} A_2 \underset{a_2}{\longrightarrow} A_3 \underset{a_3}{\longrightarrow} \cdots$$

in a category $\mathbf{C}$ is the colimit of the diagram of shape $\mathcal{X} : \omega \to \mathbf{C}$ which sends each $n \in \mathbb{N}$ to A_n. In particular, the direct limit of the A_i is an object A_∞ and morphisms $u_n : A_n \to A_\infty$ such that $u_{n+1} \circ a_n = u_n$ for each n (universal with this property). In this case, we write $A_\infty = \varinjlim X_j$.

One example of these is the direct limit in **Groups** of the groups

$$G_0 \underset{g_0}{\longrightarrow} G_1 \underset{g_1}{\longrightarrow} G_2 \underset{g_2}{\longrightarrow} G_3 \underset{g_3}{\longrightarrow} \cdots$$

which one can prove always exists. However, in my opinion, the most striking example is the direct limit of the sequence (in **Fields**):

$$\mathbb{F}_{p^{n_0}} \hookrightarrow \mathbb{F}_{p^{n_1}} \hookrightarrow \mathbb{F}_{p^{n_2}} \hookrightarrow$$

where $n_i \mid n_{i+1}$ for each i. This in, in fact, the algebraic closure of $\mathbb{F}_p$.

Definition 3.10 (Inverse Limits)**.** The *inverse limit* (or *projective limit*) of a sequence of objects and morphisms

$$A_0 \underset{a_0}{\longleftarrow} A_1 \underset{a_1}{\longleftarrow} A_2 \underset{a_2}{\longleftarrow} A_3 \underset{a_3}{\longleftarrow} \cdots$$

in a category $\mathbf{C}$ is the limit of the diagram of shape $\mathcal{X} : \omega^{\text{op}} \to \mathbf{C}$ which sends each $n \in \mathbb{N}$ to A_n. In particular, the inverse limit of the A_i is an object A_0 and morphisms $u_n : A_\infty \to A_n$ such that $u_n \circ a_{n+1} = a_n$ (universal with this property). In this case, we write $A_\infty = \varprojlim X_j$.

For example, let p be a prime and consider the sequence of rings (in **Rings**):

$$\mathbb{Z}/p\mathbb{Z} \xleftarrow[z_1]{} \mathbb{Z}/p^2\mathbb{Z} \xleftarrow[z_2]{} \mathbb{Z}/p^3\mathbb{Z} \xleftarrow[z_3]{} \cdots$$

The inverse limit, $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$, is called the *ring of p-adic integers.* As an additive group which is the inverse limit of finite groups, it is an example of a *profinite group.*

4 Constructions in Categories

Limits and colimits can be used to construct many different objects in categories. These constructions generalize common and natural concepts in specific categories.

4.1 Products and Coproducts

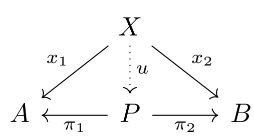
Definition 4.1 (Product)**.** Given a class of objects $P \subseteq \text{Ob}(\mathbf{C})$ in a category $\mathbf{C}$, we can form a discrete category $\mathbf{D} = \mathbf{Dis}(P)$. Define $\mathcal{X}$ to be the natural diagram of shape $\mathbf{D}$ in $\mathbf{C}$ (simply given by sending the objects to themselves). The limit of this diagram is called the *product* of the objects P_i for each $i \in \mathbf{D}$.

Definition 4.2 (Coproduct)**.** The colimit of the same diagram $\mathcal{X} : \mathbf{D} \to \mathbf{C}$ is called the *coproduct* of the objects P_i for each $i \in \mathbf{D}$.

In the case of two objects, we can define the product and coproduct more intuitively:

Proposition 8 (Alternative Definition of Product)**.** *A product diagram for the objects A and B consists of an object P and morphisms $A \xleftarrow{\pi_1} P \xrightarrow{\pi_2} B$ satisfying the following universal mapping property:*

Given any diagram $A \xleftarrow{x_1} X \xrightarrow{x_2} B$, there exists a unique $u : X \to P$ such that the following diagram commutes:



We denote the product of A and B as $A \times B$.

Proposition 9 (Alternative Definition of Coproduct)**.** *A coproduct diagram for the objects A and B consists of an object Q and morphisms $A \xrightarrow{q_1} Q \xleftarrow{q_2} B$ satisfying the following universal mapping property:*

Given any Z and $A \xrightarrow{z_1} Z \xleftarrow{z_2} B$ there is a unique $u : Q \to Z$ with $u \circ q_1 = z_1$ and $u \circ q_2 = z_2$. We denote the coproduct of A and B as $A + B$.

For example, in **Sets**, the product is the Cartesian product and the coproduct is the disjoint union of sets. In **Groups**, the product is called "the direct product of groups". We can even form products in **Cat** (where we simply define everything componentwise). In particular, the terminal and initial objects are empty products and coproducts, respectively.

Proposition 10. *There are isomorphisms $A \times (B \times C) \cong A \times B \times C \cong (A \times B) \times C$ and similarly $A + (B + C) \cong A + B + C \cong (A + B) + C$.*

Definition 4.3. A category $\mathbf{C}$ is said to have all finite products if it has a terminal object and all binary products (and therefore products of any finite cardinality). The category $\mathbf{C}$ has all (small) products if every set of objects in $\mathbf{C}$ has a product.

Definition 4.4. A *biproduct* is both a product and a coproduct.

4.2 Equalizers and Coequalizers

Definition 4.5 (Equalizers)**.** For any $f, g : A \to B$ in a category $\mathcal{C}$, the *equalizer* of f and g is an object E and morphism $e : E \to A$ such that $f \circ e = g \circ e$ with the following universal property:

Given any $z : Z \to A$ such that $f \circ z = g \circ z$, there is a unique $u : Z \to E$ such that $z = e \circ u$. In particular, this demonstrates that the morphism part of a equalizer is monic.

For example, in **Sets**, the equalizer of two functions $f, g : A \to B$ is the set $E = \{x \in A \mid f(x) = g(x)\}$ and the inclusion $E \hookrightarrow A$. In **Groups**, the equalizer of the homomorphisms $\phi, 0_{GH} : G \to H$ (where 0 is the zero homomorphism $G \to H$) is the kernel $\ker \phi$ and the inclusion $\ker \phi \hookrightarrow G$.

Definition 4.6 (Coequalizers)**.** For any parallel arrows $f, g : A \to B$ in a category, the *coequalizer* of f and g is an object Q and morphism $q : B \to Q$ such that $q \circ f = q \circ g$ with the following universal property:

Given any $z : B \to Z$ such that $z \circ f = z \circ g$, there is a unique $u : Q \to Z$ such that $z = u \circ q$. In particular, this demonstrates that any morphism of a coequalizer is epic.

Just as equalizers are generalizations of kernels, coequalizers are generalizations of equivalence relations. Indeed, in **Sets**, the coequalizer of two functions $f, g : A \to B$ is the quotient $B/\sim$ and the projection $B \to B/\sim$, where $\sim$ is the smallest equivalence relation such that $f(x) \sim g(x)$ for all $x \in A$.

In particular, if R is an equivalence relation on a set Y and r_1, r_2 are the projections $Y \times Y \supseteq R \to Y$, then the coequalizer of r_1 and r_2 is the quotient set Y/R. This can be used to provide another perspective on the classic result that the quotient of a group G by a subgroup $H \leq G$ is well-defined if and only if H is normal (i.e. the kernel of some homomorphism).

Proposition 11 (An Alternative Definition for Equalizers and Coequalizers)**.** *Let* $\mathbf{J}$ *be the category:*

$$0 \underset{g}{\overset{f}{\rightrightarrows}} 1$$

Then for any category $\mathbf{C}$*, the limit (resp. colimit) of the diagram of shape* $\mathscr{X} : \mathbf{J} \to \mathbf{C}$ *(if it exists) is the equalizer (resp. coequalizer) of the morphisms* $X_F = \mathscr{X}(f)$ *and* $X_G = \mathscr{X}(g)$.

An example of the use of coequalizers is defining "presentations" of groups. Suppose we have the group $G = \{g \mid g^8 = 1\}$. To construct G, let $F(g)$ represent the free group on g and $F(n)$, for some $n \in \mathbb{N}$, represent the free group on n elements. Then find the coequalizer of the following diagram:

$$F(1) \underset{1}{\overset{g^8}{\rightrightarrows}} F(g)$$

The result will be the group G and a morphism $F(g) \to G$ (a quotient map). For n_1 relations $(r_1 = s_1, r_2 = s_2, \ldots, r_{n_1} = s_{n_1})$ on n_2 generators $(g_1, g_2, \ldots, g_{n_2})$, the desired group will be the object resulting from the coequalizer of the following diagram:

$$F(n_1) \underset{[s_1,\ldots,s_{n_1}]}{\overset{[r_1,\ldots,r_{n_1}]}{\rightrightarrows}} F(g_1, \ldots, g_{n_2})$$

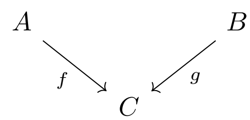
The category **Cat** is extraordinarily well-behaved: it has an initial object 0 (the empty category), a terminal object 1 (the category of one object and only the identity morphism), and all limits and colimits (thus all products, coproducts, equalizers, and coequalizers).

4.3 Pullbacks and Pushouts

Definition 4.7 (Pullback)**.** Given a category **C** and morphisms $f : A \to C$ and $g : B \to C$, the *pullback* or *fiber product* of f and g consists of morphisms $p_1 : P \to A$ and $p_2 : P \to B$ such that $f \circ p_1 = g \circ p_2$ with the following universal property:

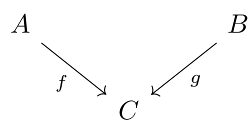
Given any $z_1 : Z \to A$ and $z_2 : Z \to B$ with $f \circ z_1 = g \circ z_2$, there exists a unique $u : Z \to P$ with $z_1 = p_1 \circ u$ and $z_2 = p_2 \circ u$.

For example, in the category **Sets**, the pullback of the diagram



is the subset $X \subseteq A \times B$ consisting of all pairs (a, b) such that $f(a) = g(b)$. Thus, it is the categorical equivalent of an equation.

Proposition 12 (An Alternative Definition for Pullbacks and Pushouts)**.** *Let* **J** *be the three-element category:*



Then, for any category **C**, *the limit of the diagram of shape* $\mathcal{X} : \mathbf{J} \to \mathbf{C}$ *is the pullback of* f *and* g *or the fiber product of* X_A *and* X_B *over* X_C.

Dually, for any category **C**, *the colimit of the diagram of shape* $\mathcal{X} : \mathbf{J} \to \mathbf{C}$ *is the pushout of* f *and* g *or the cofiber coproduct of* X_A *and* X_B *over* X_C.

Theorem 13. *For a given category* **C**, *the following are equivalent:*

1. **C** *has all finite limits.*
2. **C** *has all finite products and equalizers.*
3. **C** *has pullbacks and a terminal object.*

Corollary 13.1. *By duality, the following are equivalent:*

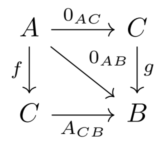
1. **C** *has all finite colimits.*
2. **C** *has all finite coproducts and coequalizers.*
3. **C** *has pushouts and an initial object.*

4.4 Zero Morphisms and Kernels

Definition 4.8 (Left, Right, and Two-Sided Zero Morphisms)**.** A morphism $f : X \to Y$ is called a *left zero morphism* if, for any $g, h : W \to X$, $f \circ g = f \circ h$. Dually, a morphism $f : X \to Y$ is called a *right zero morphism* if, for any $g, h : Y \to Z$, $g \circ f = h \circ f$. If a morphism is both a left and a right zero morphism, we call it a *(two-sided) zero morphism.*

For example, the zero homomorphism $0_{GH} : G \to H$ is a zero morphism for every group $G, H \in$ **Groups**. There are no left zero morphisms in **Sets** and the only right zero morphisms are the functions $f : \varnothing \to S$ for arbitrary sets S.

Definition 4.9 (Categories With Zero Morphisms)**.** We say a category $\mathbf{C}$ has zero morphisms if, for every object $A, B \in \mathbf{C}$, there is a fixed zero morphism $0_{AB} \to \mathbf{C}$ such that, for all objects C and morphisms $f : A \to C$ and $g : C \to B$, the following diagram commutes:



Definition 4.10 (Kernel and Cokernel)**.** Suppose that $\mathbf{C}$ is a category with zero morphisms. Then if $f : X \to Y$ is an arbitrary morphism in $\mathbf{C}$, a *kernel* of f is an equalizer of f and 0_{XY}. Dually, a *cokernel* of f is a coequalizer of f and 0_{XY}.

We let $\mathrm{Ker}(f)$ denote the object associated with this equaliser and $\ker(f)$ denote the morphism. The notation is similar for cokernels: $\mathrm{Coker}(f)$ is the object and $\mathrm{coker}(f)$ is the morphism. Recall that because they are equalizers and coequalizers respectively, $\ker(f)$ is a monomorphism and $\mathrm{coker}(f)$ is an epimorphism for any f (when they are defined).

Lemma 14. *Let $\mathbf{C}$ be a category and $Z \in \mathrm{Ob}(\mathbf{C})$ is a zero object. Then a morphism $\zeta : A \to B$ is a zero morphism if and only if it factors through Z – i.e. if $\zeta = \beta \circ \alpha$ where $\alpha : A \to Z$ and $\beta : Z \to B$ are the unique morphisms guaranteed by Z being a zero object.*

Proof. For the $\Leftarrow$ direction, consider an arbitrary object C, arbitrary morphisms $\phi, \xi : C \to A$, and suppose that $\zeta = \beta \circ \alpha$. Then $\alpha \circ \phi$ and $\alpha \circ \xi$ are morphisms $C \to Z$ − by uniqueness they must be equal. Then by composing β on the left, $\zeta \circ \phi = \zeta \circ \xi$ so ζ is a left zero morphism. By dual logic, ζ is a right zero morphism.

For the $\Rightarrow$ direction, let $\zeta : A \to B$ be a zero morphism and β' be the unique morphism $B \to Z$. Applying the zero morphism property to 1_B and $\beta \circ \beta'$ demonstrates that $\zeta = \beta \circ \beta' \circ \zeta$. But notice that $\beta' \circ \zeta$ is a morphism $A \to Z$, so by uniqueness it is α. Thus $\zeta = \beta \circ \alpha$. □

4.5 Images and Coimages

Definition 4.11 (Image and Coimage)**.** Let $\mathbf{C}$ be a category and $\phi : A \to B$ a morphism in $\mathbf{C}$. Then the kernel of the cokernel of ϕ is called the *image* of ϕ. In particular, $\mathrm{im}(\phi)$ is the morphism $\ker(\mathrm{coker}(\phi))$ and $\mathrm{Im}(\phi)$ is the object $\mathrm{Ker}(\mathrm{Coker}(\phi))$.

Dually, the *coimage* of ϕ is the cokernel of the kernel of ϕ. We use the obvious notation $\mathrm{coim}(\phi)$ and $\mathrm{Coim}(\phi)$ for the morphism and object of this resulting construction.

5 Exponentials and Yoneda's Lemma

Consider a function of sets $f(x, y) : A \times B \to C$ written using $x \in A$ and $y \in B$. If we hold a fixed, we are given a function $f(a, y) : B \to C \in C^B$. Thus the original function f induces a map $\overline{f} : A \to C^B$ given by $a \mapsto f(a, y)$. Notice that $\overline{f}$ is uniquely determined by the requirement $\overline{f}(a)(b) = f(a, b)$.

Indeed, any map $\phi : A \to C^B$ is of the form $\overline{f}$ for some function $f : A \times B \to C$, since we can set $f(a, b) := \phi(a)(b)$. Thus, $\mathrm{Hom}_{\mathbf{Sets}}(A \times B, C) \cong \mathrm{Hom}_{\mathbf{Sets}}(A, C^B)$.

There is another important property of sets of functions: there is an evaluation function $\epsilon : C^B \times B \to C$ given by $(f, b) \mapsto f(b)$. This function has the universal mapping property that, for any $f : A \times B \to C$, there is a unique $\overline{f} : A \to C^B$ such that $\epsilon \circ (\overline{f}, 1_B) = f$ – in other words $\epsilon(\overline{f}(a), b) = f(a, b)$.

All of this inspires a categorical generalization: the exponential.

5.1 Exponentials

Definition 5.1. Let **C** be a category with binary products. An *exponential* of objects B and C consists of an object C^B and a morphism $\epsilon : C^B \times B \to C$ such that, for any object A and morphism $f : A \times B \to C$, there is a unique $\overline{f} : A \to C^B$ that makes the following diagram commute:

$$\begin{array}{ccc} C^B \times B & \xrightarrow{\epsilon} & C \\ {\scriptstyle \overline{f}\times 1_B}\uparrow & \nearrow_{f} & \\ A \times B & & \end{array}$$

In this case, ϵ is called the *evaluation* morphism and $\overline{f}$ is called the *transpose* morphism of f. Furthermore, the transposition operation

$$(f : A \times B \to C) \to (\overline{f} : A \to C^B)$$

is an inverse to the operation

$$(g : A \to C^B) \to (\overline{g} = \varepsilon \circ (g \times 1_B) : A \times B \to C)$$

giving us the isomorphism $\mathrm{Hom}_{\mathcal{C}}(A \times B, C) \cong \mathrm{Hom}_{\mathcal{C}}(A, C^B)$.

Definition 5.2 (Cartesian Closed)**.** A category with all finite products and exponentials is called *cartesian closed.*

For example, **Sets** is a cartesian closed, as C^B is simply the set of functions from $B \to C$ and the morphism $\epsilon : C^B \times B \to C$ is just the standard evaluation morphism $(b, f) \to f(b)$.

Proposition 15. *The category* **Cat** *is cartesian closed, with the exponentials being* $\mathbf{D}^{\mathbf{C}} = \mathbf{Fun}(\mathbf{C}, \mathbf{D})$.

In particular, $\mathbf{C}^{\mathbf{1}} = \mathbf{C}$ and $\mathbf{C}^{\mathbf{2}} = \mathbf{C}^{\to}$, the morphism category. Another example: for any set I regarded as a discrete category, we have a isomorphism of categories $\mathbf{C}^I = \prod_{i \in I} \mathbf{C}$.

5.2 Presheaves, Sheaves, and Yoneda's Lemma

Definition 5.3 (Presheaves in Topology)**.** Take a topological space X and a category **C**. Then a *presheaf* $\mathcal{F}$ on X is functor with values in **C** with the following data:

1. Every open set $U \subseteq X$ corresponds to an object $\mathcal{F}(U) \in \mathrm{Ob}(\mathbf{C})$.
2. Every inclusion of open sets $V \subseteq U$ corresponds to a morphism $\mathrm{res}_{V,U} : F(U) \to F(V)$ in **C**.

Of course, we require the morphisms (called *restriction morphisms*) to satisfy the following properties:

1. $\mathrm{res}_{U,U} : F(U) \to F(U)$ is the identity morphism on $F(U)$.
2. For any three open sets $W \subseteq V \subseteq U$, $\mathrm{res}_{W,V} \circ \mathrm{res}_{V,U} = \mathrm{res}_{W,U}$.

If $\mathcal{F}$ is a **C**-valued presheaf on X and U is an open subset of X, then $F(U)$ is called the *sections of* $\mathcal{F}$ *over* U. If **C** is a concrete category, then each element of $F(U)$ is called a *section.* In particular, if s is a section of $F(U)$, then $\mathrm{res}_{V,U}(s)$ is denoted $s|_V$. Finally, a section over X is called a *global section.*

Definition 5.4 (Presheaves in Category Theory)**.** A *presheaf* on a category **C** is a contravariant functor $\mathbf{C} \to \mathbf{Sets}$ (equivalently, it is a covariant functor $\mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$). Thus, the category of all presheaves on a category **C** is denoted $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$.

In particular, if **C** is the poset of open sets in a topological space X interpreted as a category, then this definition coincides with the earlier definition of presheaves in a topology.

Definition 5.5 (Sheaves in Topology)**.** A presheaf to **Sets** is a *sheaf* it is satisfies the following two axioms:

1. If (U_i) is an open covering of an open set U and if $s, t \in F(U)$ such that $s|U_i = t|U_i$ for each U_i in the covering, then $s = t$.
2. If (U_i) is an open covering of an open set U and if, for each i, a section $s_i \in F(U_i)$ is given such that for each pair U_i, U_j, $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ (they agree on the shared spaces), then there is a section $s \in F(U)$ such that $s|_{U_i} = s_i$ for each i.

We call the first axiom the *locality axiom* and the second axiom the *gluing axiom.*

In short, sheaves are generalizations of normal topological spaces. They retain structure as images of topological spaces, but can be strictly more free. For example, if we let our topological space be $X = \mathbb{R}^n$, then on each open set $U \subseteq$ we have the ring of differentiable functions $\mathcal{O}(U)$. This association induces a sheaf from X to **Sets**, and is the prototypical example of a sheaf – yet it is not immediately clear how one would make the set of differentiable functions on X into a topological space.

Definition 5.6 (Embeddings)**.** A functor $\mathcal{F} : C \to D$ is called an *embedding* if it is full, faithful, and injective on objects.

Definition 5.7 (The Yoneda Embedding)**.** The *Yoneda embedding* is the functor $\mathscr{Y} : \mathbf{C} \to \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ taking $C \in \mathbf{C}$ to the representable functor

$$\mathscr{Y}(C) = \mathrm{Hom}_{\mathbf{C}}(-, C) : \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$$

and taking $f : C \to D$ to the natural transformation

$$\mathscr{Y}(f) = \mathrm{Hom}_{\mathbf{C}}(-, f) : \mathrm{Hom}_{\mathbf{C}}(-C,) \to \mathrm{Hom}_{\mathbf{C}}(-D,)$$

It is not immediately obvious that the Yoneda embedding is actually an embedding, but we will eventually prove that this is the case.

One should thus think of the Yoneda embedding as a "representation" of C in a category of set-valued functors and natural transformations on some index category.

Lemma 16 (Yoneda's Lemma)**.** *Let* $\mathbf{C}$ *be locally small. For any object* $C \in \mathbf{C}$ *and functor* $\mathcal{F} \in \mathbf{Sets}^{\mathbf{C}^{op}}$*, there is an isomorphism*

$$\mathrm{Hom}_{\mathbf{Sets}^{\mathbf{C}^{op}}}(\mathscr{Y}(C), \mathcal{F}) \cong \mathcal{F}(C)$$

which is natural in both $\mathcal{F}$ *and* C*. More explicitly, the naturality in* $\mathcal{F}$ *implies that the following diagram commutes for any* $\mathscr{X} : \mathcal{F} \to \mathcal{G}$*:*

$$\begin{array}{ccc} \mathrm{Hom}(\mathscr{Y}(C), \mathcal{F}) & \xrightarrow{\cong} & \mathcal{F}(C) \\ {\scriptstyle \mathrm{Hom}(\mathscr{Y}(C), \mathscr{X})}\big\downarrow & & \big\downarrow{\scriptstyle \mathscr{X}} \\ \mathrm{Hom}(\mathscr{Y}(C), \mathcal{G}) & \xrightarrow[\cong]{} & \mathcal{G}(C) \end{array}$$

Similarly, the naturality in C *implies that the following diagram commutes for any* $h : C \to D$*:*

$$\begin{array}{ccc} \mathrm{Hom}(\mathscr{Y}(C), \mathcal{F}) & \xrightarrow{\cong} & \mathcal{F}(C) \\ {\scriptstyle \mathrm{Hom}(\mathscr{Y}(h), \mathcal{F})}\big\downarrow & & \big\downarrow{\scriptstyle \mathscr{X}} \\ \mathrm{Hom}(\mathscr{Y}(D), \mathcal{F}) & \xrightarrow[\cong]{} & \mathcal{F}(D) \end{array}$$

Proof. Covered in pages 189-192 of Awodey's Category Theory. □

Theorem 17. *The Yoneda embedding* $\mathscr{Y} : \mathbf{C} \to \mathbf{Sets}^{\mathbf{C}^{op}}$ *is full and faithful.*

Also observe that $\mathscr{Y}$ is injective on objects. For, given objects A, B in $\mathbf{C}$, if $\mathscr{Y}(A) = \mathscr{Y}(B)$ then $1_A \in \mathrm{Hom}(A, A) = \mathscr{Y}(A)(A) = \mathscr{Y}(B)(A)$. Thus the Yoneda embedding is actually an embedding.

5.3 Applications of Yoneda's Lemma

Corollary 17.1. *Given objects A and B in a locally small category $\mathbf{C}$, $\mathscr{Y}(A) \cong \mathscr{Y}(B)$ if and only if $A \cong B$. In particular, we can think of this by saying that an object is determined exactly by its relationships with other objects.*

Corollary 17.2. *In a cartesian closed category $\mathbf{C}$, $(A^B)^C \cong A^{(B\times C)}$.*

Corollary 17.3. *If a cartesian closed category $\mathbf{C}$ also has coproducts, then $A \times (B + C) \cong (A \times B) + (A \times C)$.*

6 Adjoints

Definition 6.1 (Adjunctions)**.** An *adjunction* consists of functors $\mathcal{F} : \mathbf{C} \to \mathbf{D}$ and $\mathcal{U} : \mathbf{D} \to \mathbf{C}$ and a natural isomorphism $\phi : \text{Hom}_{\mathbf{D}}(\mathcal{F}(C), D) \xrightarrow{\sim} \text{Hom}_{\mathbf{C}}(C, \mathcal{U}(D))$.

Definition 6.2 (Adjoint)**.** In this case, $\mathcal{F}$ is called the *left adjoint* and $\mathcal{U}$ is called the *right adjoint.* One writes $\mathcal{F} \dashv \mathcal{U}$ say $\mathcal{F}$ and $\mathcal{U}$ are *adjoint functors.*

Definition 6.3 (Units and Counits)**.** We call $\eta : 1_{\mathbf{C}} \to \mathcal{U} \circ \mathcal{F}$ determined by $\eta_C = \phi(1_{\mathcal{F}(C)})$ for any $C \in \mathbf{C}$ the *unit* of the adjunction. Similarly, we call $\epsilon : \mathcal{F} \circ \mathcal{U} \to 1_{\mathbf{D}}$ determined by $\varepsilon_D = \xi(1_{\mathcal{U}(D)})$ for any $D \in \mathbf{D}$ the *counit* of the adjunction.

Proposition 18. *Adjoins are unique up to isomorphism. Specifically, given a functor $\mathcal{F} : \mathbf{C} \to \mathbf{D}$ and two other functors $U, V : \mathbf{D} \to \mathbf{C}$ such that $F \dashv U$ and $F \dashv V$, we have that $U \cong V$.*

A particularly exciting example are the adjoint functors between the orbit category of G and the poset category of intermediate fields between k and a field extension F/k – more commonly known as the Galois correspondence described by the Fundamental Theorem of Galois Theory. In fact, this is often generalized: a *Galois connection* is an adjunction between functors on posets. To see more examples, visit this link.

7 The Fundamental Group as a Functor

In this section, we will develop the necessary tools to prove Brouwer's Fixed-Point Theorem using functors and the fundamental group. Since we are working within $\mathbb{R}^n$, one only needs to know basic definitions: what a subset is, the Euclidean metric, etc.

7.1 The Necessary Prerequisites

Definition 7.1 (Space)**.** A *space* X is a subset of $\mathbb{R}^n$. A *pointed space* is a pair (X, x_0) of a space X and a point $x_0 \in X$, called the *basepoint of X*.

For example, the space $\mathbb{R}^n$ is called *Euclidean n-space.* $\mathbb{R}^0$ - a single element - is called a *point.* We define the *n-disk* as

$$D_n := \{x \in \mathbb{R}^n \mid |x| \leq 1\}.$$

Similarly, we define the *n-sphere* as

$$S_n := \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}.$$

Finally, the *unit interval* is simply the subset $I := [0, 1] \in \mathbb{R}$.

Definition 7.2 (Continuity)**.** A map of spaces $f : X \to Y$ is *continuous* if, for any sequence of points $\{x_i\}_{i\in\mathbb{N}}$ converging to $x \in X$, the sequence $\{f(x_i)\}_{i\in\mathbb{N}}$ converges to $f(x) \in Y$. In particular, a continuous map of pointed spaces $f : (X, x_0) \to (Y, y_0)$ is a continuous map of spaces $f : X \to Y$ such that $f(x_0) = y_0$.

Theorem 19. *A map $f : S \to \mathbb{R}^n$ given by $s \mapsto (f_1(s), f_2(s), \ldots, f_n(s))$ is continuous if and only if $s \mapsto f_i(s)$ is a continuous map $S \to \mathbb{R}$ for each $i \in \{1, \ldots, n\}$.*

Definition 7.3 (Paths and Loops)**.** A *path* in a space X is a continuous map $f : I \to X$. A *loop* in a pointed space (X, x_0) is a path $f : I \to X$ such that $f(0) = f(1) = x_0$.

Definition 7.4 (Homotopies of Paths and Loops)**.** A *homotopy of paths* on X is a continuous map $f : I \times I' \to X$ with $f(0, t) = f(0, 0)$ and $f(1, t) = f(1, 0)$ for all t. A *homotopy of loops* on (X, x_0) on (X, x_0) is a continuous map $f : I \times I' \to X$ with $f(0, t) = f(1, t) = x_0$.

Definition 7.5 (Homotopic and Nullhomotopic)**.** Define $f_t : I \to X$ as the path $s \mapsto f(s, t)$. Two paths g and h are *homotopic* (denoted $g \sim h$) if there exists a homotopy $f : I \times I' \to X$ such that $f_0 = g$ and $f_1 = h$. Furthermore, a loop is *nullhomotopic* if it is homotopic to the constant loop (i.e. the loop $f : I \to X$ given by $f(t) = x_0$ for all t).

For example, given the pointed space $(\mathbb{R}^2, (0, 0))$, the loop $f : I \to \mathbb{R}^2$ given by $s \mapsto (1 - \cos(2\pi s), \sin(2\pi s)$ is nullhomotopic (as we squash the circle into a point).

Proposition 20. *Homotopy defines an equivalence relation on the set of loops in a space: in particular, it is reflexive, symmetric, and transitive.*

Proof. The constant homotopy proves that this relation is reflexive. Symmetry follows from replacing t with $1 - t$ (doing the interpolation in reverse). Finally, transitivity comes from doing the homotopy $f \sim g$ and $g \sim h$ at double speed and linking them together. More explicitly, if F is a homotopy between f and g and G is a homotopy between g and h,

$$H(s, t) = \begin{cases} F(s, 2t) & \text{if } 0 \le t \le \frac{1}{2} \\ G(s, 2t - 1) & \text{if } \frac{1}{2} < t \le 1 \end{cases}$$

is a homotopy from f to h. □

Thus, in particular, the equivalence relation of homotopy partitions the loops (X, x_0) into equivalence classes called *homotopy classes*.

7.2 The Fundamental Group

Definition 7.6 (Composition of Paths and Loops)**.** Let $f, g : I \to X$ be two paths. Then the composition of f and g, $f \star g : I \to X$ is defined as

$$(f \star g)(t) := \begin{cases} f(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ g(2t - 1) & \text{if } \frac{1}{2} < t \le 1 \end{cases}$$

This is also well-defined for loops: if f and g are loops, then $f \star g$ will also be a loop. Furthermore, this respects the equivalence relation of homotopy: it should be clear that if $f_0 \sim f_1$ and $g_0 \sim g_1$, then $f_0 \star g_0 \sim f_1 \star g_1$.

Definition 7.7 (Fundamental Groups)**.** The *fundamental group* of (X, x_0), denoted $\pi_1(X, x_0)$ is the group whose underlying set is the set of loops of (X, x_0) up to homotopy, with composition operation given by $[f] \cdot [g] = [f \star g]$ for loops $f, g : I \to X$.

Proposition 21. *The fundamental group $\pi_1(X, x_0)$ is a group.*

Proof. The identity is given by the constant path $e : I \to X$ sending $t \mapsto x_0$. Given a loop $f : I \to X$, the inverse loop $f^{-1} : I \to X$ is $f^{-1}(t) = f(1 - t)$. As it turns out, though $f \star f^{-1} \ne e$, they are homotopic (to see this, move the midpoint of the path backwards along the path until it coincides with the constant path). Finally, associativity holds because clearly $[(f \star g) \star h] = [f \star (g \star h)]$. □

Definition 7.8 (Path Connected and Simply Connected)**.** A space X is *path connected* if there is a path joining any two points. A space is *simply connected* if it is path connected and, for all points $x \in X$, $\pi_1(X, x) = 0$ (that is, any loop can be reduced to the constant path).

Lemma 22. *If X is a path connected space, then $\pi_1(X, x) \cong \pi_1(X, y)$ for any two points $x, y \in X$.*

Definition 7.9 (Homeomorphisms)**.** A continuous map of spaces $f : X \to Y$ is called a *homeomorphism* if there is a map $g : Y \to X$ with $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. In this case, we say X and Y are homeomorphic and write $X \cong Y$.

Theorem 23. *If $f : X \to Y$ is a homeomorphism with $x \to y$, then $\pi_1(X, x) \cong \pi_1(Y, y)$.*

Proof. Left as an exercise. □

7.3 A Quick Aside: Topological Groups

Definition 7.10 (Group Spaces and Topological Groups)**.** A *group space* (or *topological group*) is a space G with a continuous multiplication map $m : G \times G \to G$ and a continuous inversion map $i : G \to G$ making the underlying set of G into a group.

Theorem 24. *If G is a group space and $e \in G$ is the identity point, then $\pi_1(G, e)$ is abelian.*

Proof. This follows immediately from the definition, but it is fairly tedious: one must actually exhibit an explicit formula. □

Corollary 24.1. *$\pi_1(S^1, x_0)$ is abelian.*

Proof. S^1 can be given a group structure by defining x_0 to be the identity and letting any point $y \in S^1$ act as the rotation given by dragging y to x_0. Thus (S^1, x_0) is group space, so the result follows. □

7.4 Proving Brouwer's Fixed-Point Theorem

Lemma 25. *The fundamental group G of (S_1, x_0) is nontrivial.*

Proof. The constant path is not homotopic to the path comprised of a single loop around the circle. Thus there are at least two homotopy classes, so $|G| > 1$. □

Proposition 26. *The map from the category of pointed topological spaces* $\mathbf{Top}_*$ *to* **Groups** *given by $(X, x_0) \to \pi_1(X, x_0)$ is a functor, which we denote π_1.*

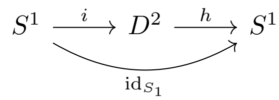
Proof. By definition of a functor, this simply amounts to showing that for any two continuous maps of pointed spaces $f : (X, x_0) \to (Y, y_0)$ and $g : (Y, y_0) \to (Z, z_0)$, the following hold:

1. f induces a homomorphism $\pi_1(f) : \pi_1(X, x_0) \to \pi_1(Y, y_0)$.
2. If f is the identity map $(X, x_0) \to (X, x_0)$, then $\pi_1(f)$ is the identity map of groups $\pi_1(X, x_0) \to \pi_1(X, x_0)$.
3. The homomorphism $\pi_1(g \circ f)$ is equal to the homomorphism $\pi_1(f) \circ \pi_1(g)$.

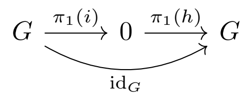
Filling in the details is left as an exercise to the reader. □

Theorem 27 (Brouwer's Fixed-Point Theorem)**.** *Any continuous map from the unit disk D^2 to itself has a fixed point.*

Proof. Assume the theorem is false by taking a continuous map $g : D^2 \to D^2$ so that $x \neq g(x)$ for all $x \in D^2$. Define a continuous map $h : D^2 \to S^1$ given by mapping x to the point on the boundary of D^2 on the ray starting at $g(x)$ and through x. Notice $h(x) = x$ for every $x \in S^1$. Then let i denote the inclusion map $S^1 \hookrightarrow D^2$, so that $h \circ i = \mathrm{id}_{S^1}$. In other words, the following diagram commutes:



Notice that any loop D^2 continuously deforms to a given point, so $\pi_1(D^2) = 0$, the trivial group. Thus, applying the functor π_1 to this diagram gives us the following diagram:



But clearly this is impossible since G is nontrivial, hence the result follows. □

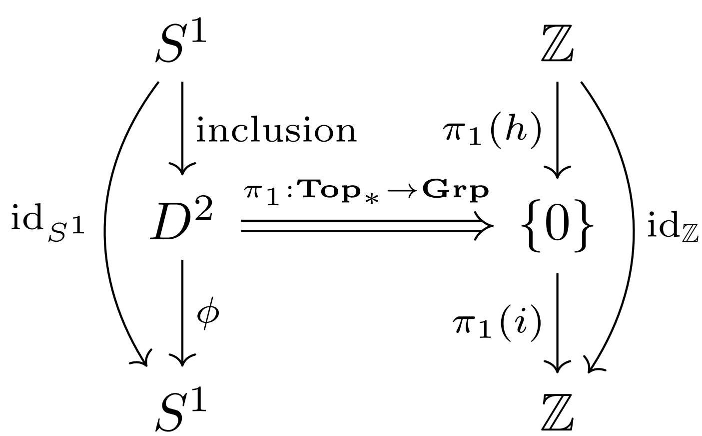
7.5 Applications of Brouwer's Fixed-Point Theorem

Maps: Consider a map of a country. If that map is placed anywhere in that country, there will always be a point on the map that represents that exact point in the country. Equivalently, given two similar maps of a country of different sizes resting on top of each other, there always exists a point that represents the same place on both maps.

Game Theory: A beautiful application of Brouwer's Fixed-Point Theorem is that the Game of Hex must always have a winner: to find out more, visit this link. The existence of Nash equilibria, a fundamental result in game theory, is also linked to Brouwer's Fixed-Point Theorem.

7.6 Example: A One-Diagram Proof

Theorem 28. *There is no continuous function $\phi : D^2 \to S^1$ fixing S^1.*



Explanation: If there was such a continuous function ϕ, then the left-hand side would be commutative. But by the functoriality of π_1, this would imply the right-hand side is commutative – clearly an impossibility since no possible map $\{0\} \to \mathbb{Z}$ is surjective.

8 An n-Category

Definition 8.1 (n-morphisms)**.** A 1-morphism is any morphism between objects in a category. Then, for any natural number $n > 1$, an *n-morphism* is a morphism between $(n-1)$-morphisms – i.e. a pair of maps g_1 and g_2 such that:

$$\begin{array}{ccc} A & \xrightarrow{g_1} & A' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle h} \\ B & \xrightarrow[g_2]{} & B' \end{array}$$

where f and h are $(n-1)$-morphisms, so A, B, A', B' are $(n-2)$-morphisms.

Definition 8.2 (n-category)**.** An *n-category* is a category with objects and morphisms plus k-morphisms for every $k \leq n$.

For an example of why a 2-category might be interesting, consider the category **Cat**. The morphisms in **Cat** are functors, so the 2-morphisms are natural transformations. Thus the 2-category version of **Cat** more naturally contains the structure of natural transformations.

But then, to study 2-categories, we need 3-categories, and so on. At the end of the road, we encounter the *∞-category*, which contains n-morphisms for every $n \in \mathbb{Z}^+$. These categories are used extensively in algebraic topology (especially homotopical algebra).

9 Applications of Category Theory and Where to Learn More

I used Awodey's book on Category Theory to learn and create these notes. Another good book is "Category Theory for the Working Mathematician" by Mac Lane, the founder of the field. If you'd like to explore, try out the following resources. They include places to learn about applied category theory, if you so desire.

1. Math3ma: Limits and Colimits, Part 1
2. Math3ma: Limits and Colimits, Part 2
3. Math3ma: Limits and Colimits, Part 3
4. Math3ma: Limits and Colimits, Part 4
5. The Fundamental Group
6. Brouwer's Fixed Point Theorem
7. Proving Brouwer's Fixed Point Theorem — Infinite Series
8. The Fundamental Group and Brouwer's Fixed Point Theorem
9. Brouwer's Fixed Point Theorem
10. Programming and Category Theory, Part 1
11. Relating Category Theory to Programming Language Theory
12. Seven Sketches in Compositionality: An Invitation to Applied Category Theory
13. What is Applied Category Theory?
14. Why Category Theory Matters