Point-Set Topology
An Introduction

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August 2021

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Introduction

These notes were compiled with Evelyn Binoya as we read through the first half of Munkres' book *Topology (2nd Ed).* The follow up to these notes are given by my notes on Algebraic Topology, which follows Hatcher.

1 General Notions

1.1 Topological Spaces

Definition 1 (Topology)**.** A *topology* on a set X is a pair $(X, \mathcal{T})$ where $\mathcal{T}$ is a collection of subsets of X such that $\varnothing$ and X are in $\mathcal{T}$, the union of an arbitrary number of elements of $\mathcal{T}$ is in $\mathcal{T}$, and the intersection of any finite number of elements of $\mathcal{T}$ is in $\mathcal{T}$. The elements of $\mathcal{T}$ are called the *open sets* of X.

When the collection of open sets $\mathcal{T}$ is clear, we will denote the topology $(X, \mathcal{T})$ by X as an abuse of notation.

Definition 2 (Neighborhood)**.** A *neighborhood of* x is an open set containing x.

Definition 3 (Comparing Topologies)**.** Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a set X. Then we say that $\mathcal{T}'$ is *finer* (resp. *strictly finer*) than $\mathcal{T}$ if $\mathcal{T} \subseteq \mathcal{T}'$ (resp. $\mathcal{T} \subsetneq \mathcal{T}'$). Similarly, we say that $\mathcal{T}'$ is *coarser* (resp. *strictly coarser*) than $\mathcal{T}$ if $\mathcal{T}' \subseteq \mathcal{T}$ (resp. $\mathcal{T}' \subsetneq \mathcal{T}$). In either case, we say $\mathcal{T}$ and $\mathcal{T}'$ are *comparable.*

Definition 4 (Examples of Topologies)**.** The following are examples of topologies on an arbitrary set X.

1. The *discrete topology* on X is given by setting $\mathcal{T} = \mathcal{P}(X)$.
2. The *trivial topology* on X is given by setting $\mathcal{T} = \{\varnothing, X\}$.
3. The *finite complement topology* (or *cofinite topology*) on X is given by setting

$$\mathcal{T} = \{U \subseteq X \mid X \setminus U \text{ is finite}\} \cup \{\varnothing\}.$$

Definition 5 (Basis For a Topology)**.** Let X be a set and $\mathcal{B}$ be a collection of subsets of X. Then $\mathcal{B}$ is a *basis for a topology on* X if the following axioms are satisfied:

1. For each $x \in X$, there exists $B \in \mathcal{B}$ containing x.
2. For each $x \in X$ contained in $B_1, B_2 \in \mathcal{B}$, there exists B_3 containing x and contained in $B_1 \cap B_2$. This property implies any finite intersection of elements of B can be expressed as a union of elements of B.

The *topology generated by a basis* $\mathcal{B}$ is defined as so: a subset $U \subseteq X$ is open if for each $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq U$. It is simple to check that this gives a valid topology on X in which every element of $\mathcal{B}$ is open.

Theorem 1 (Other Descriptions of Bases)**.** *Let X be a set with topology $\mathcal{T}$. Then, if $\mathcal{B}$ is a collection of subsets of X, the following propositions are equivalent:*

1. *$\mathcal{B}$ is a basis generating the topology $\mathcal{T}$.*
2. *$\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{B}$.*
3. *$\mathcal{B}$ is a collection of open sets satisfying the following property: for each open set U of X and each $x \in U$, there is an element $B \in \mathcal{B}$ such that $x \in B \subseteq U$.*

Proof. First, we will prove that (1) implies (2). Now, clearly if $\mathcal{T}$ is the topology generated by the basis $\mathcal{B}$, then $\mathcal{B} \subseteq \mathcal{T}$. Therefore, the collection of all unions of elements of $\mathcal{B}$ is contained in $\mathcal{T}$ (by definition of a topology). Conversely, given $U \in \mathcal{T}$, choose for each $x \in U$ an element B_x of $\mathcal{B}$ such that $x \in B_x \subseteq U$. Then $U = \bigcup_{x \in U} B_x$, so U equals a union of elements of $\mathcal{B}$.

Next, we will prove that (2) implies (1). First, we show that $\mathcal{B}$ is a basis. To see why the first property holds, notice that because $\mathcal{T}$ contains X, the union of some collection of elements of $\mathcal{B}$ must be X. Therefore, for any point $x \in X$, there must exist an element $B \in \mathcal{B}$ containing x. For the second property, notice that because $\mathcal{T}$ is closed under finite intersections, $\mathcal{B}_1 \cap \mathcal{B}_2 = \bigcup_{\lambda \in \Lambda} B_\lambda$ for some collection of $B_\lambda \in \mathcal{B}$. Then, if $x \in \mathcal{B}_1 \cap \mathcal{B}_2$, there exists B_3 containing x; clearly B_3 is also contained in $\mathcal{B}_1 \cap \mathcal{B}_2$, as desired. Now that we have shown $\mathcal{B}$ is a basis, it suffices to show that $\mathcal{T}$ is the topology generated by $\mathcal{B}$. Yet now that we know that $\mathcal{B}$ is a basis, the proof for this is identical to the proof given in the section (1) $\Rightarrow$ (2).

Hence we have proven that (1) and (2) are equivalent. Now, the fact that (1) implies (3) is trivial, so assume (3). First, we must show that $\mathcal{B}$ is a basis. The first condition for a basis is easy: X is an open set, so there an element $B \in \mathcal{B}$ such that $x \in B$. For the second condition, consider $B_1, B_2 \in \mathcal{B}$ and take an arbitrary point $x \in B_1 \cap B_2$. Yet $B_1 \cap B_2$ is an open set, so there exists $B_3 \subseteq B_1 \cap B_2$ containing x, as desired.

Finally, we must show that the topology $\mathcal{T}'$ generated by $\mathcal{B}$ equals the topology $\mathcal{T}$. Now, if $U \in \mathcal{T}$ and $x \in U$, then by hypothesis there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$. Therefore, U belongs to the topology $\mathcal{T}'$. Conversely, if $V \in \mathcal{T}'$, then V equals a union of elements of $\mathcal{B}$ by our above work. Since each element of $\mathcal{B}$ is in $\mathcal{T}$, V must also be in $\mathcal{T}$. Hence we are done. □

Definition 6 (Standard Topology on $\mathbb{R}$)**.** Let $\mathcal{B}$ be the collection of all open intervals in $\mathbb{R}$; that is, all sets of the form $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$. Then $\mathcal{B}$ is a basis of a topology, called the *standard topology on* $\mathbb{R}$.

Lemma 2 (Comparing Topologies With Bases)**.** *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on a set X. Then $\mathcal{T}'$ is finer than $\mathcal{T}$ if and only if the following property holds:*

For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.

Proof. Assume $\mathcal{T}'$ is finer than $\mathcal{T}$. Take $x \in X$ and $B \in \mathcal{B}$ with $x \in B$ arbitrarily. Since B belongs to $\mathcal{T}$ and $\mathcal{T} \subseteq \mathcal{T}'$, we must have $B \in \mathcal{T}'$. But then by definition of the topology generated by a basis, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$, as desired.

Conversely, suppose that for each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$. Then take an element $U \in \mathcal{T}$ arbitrarily. Take $x \in U$ arbitrarily. Since $\mathcal{B}$ generates $\mathcal{T}$, there is an element $B \in \mathcal{B}$ such that $x \in B \subseteq U$ by definition. But then there exists an element $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$. Then $x \in B' \subseteq U$, so $U \in \mathcal{T}'$, as desired. □

Definition 7 (Subbasis)**.** A *subbasis* $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X. The *topology generated by the subbasis* $\mathcal{S}$ is defined to be the collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$.

1.2 Closed Sets

Definition 8 (Closed Sets)**.** Let X be a topological space. Then $A \subseteq X$ is called *closed* if $X \setminus A$ is open.

Theorem 3 (Properties of Closed Sets)**.** *Let X be a topological space. Then the following conditions hold:*

1. *$\varnothing$ and X are closed.*
2. *Arbitrary intersections of closed sets are closed.*
3. *Finite unions of closed sets are closed.*

In fact, this definition can be used to define a topological space: by distinguishing closed sets and then defining open sets to be the complement of closed sets.

Definition 9 (Interior and Closure)**.** Let A be a subset of a topological space X. Then the *interior* of A is defined as the union of all open sets contained in A and denoted $\operatorname{Int} A$ or A°. The *closure* of A is defined as the intersection of all closed sets containing A, and is denoted $\operatorname{Cl} A$ or $\overline{A}$.

Theorem 4 (Alternative Definitions of Closures)**.** *Let A be a subset of the topological space X.*

1. *Then $x \in \overline{A}$ if and only if every neighborhood of x intersects A.*
2. *If X has basis $\mathcal{B}$, then $x \in \overline{A}$ if and only if every basis element $B \in \mathcal{B}$ containing x intersects A.*

1.3 Continuous Functions and Homeomorphisms

Definition 10 (Continuous Functions)**.** A function $f : X \to Y$ is said to be *continuous* if, for each open subset $V \subseteq Y$, the preimage $f^{-1}(V)$ is an open subset of X. Equivalently, if $\mathcal{B}$ is a basis for the topology on Y, then f is continuous if the preimage of each basis element $B \in \mathcal{B}$ is open in X.

Theorem 5 (Equivalent Conditions for Continuity)**.** *Let $f : X \to Y$ be a map of topological spaces. Then the following are equivalent:*

1. *f is continuous.*
2. *For every subset A of X, one has $f(\overline{A}) \subseteq \overline{f(A)}$.*
3. *For every closed set B of Y, the set $f^{-1}(B)$ is closed in X.*
4. *For each $x \in X$ and each neighbhorhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subseteq V$.*

Note that if (4) holds for any select $x \in X$, we say that f is continuous at the point x.

Proof. (1) $\Rightarrow$ (2). Let A be a subset of X. Take $x \in \overline{A}$. Then, if V is a neighborhood of $f(x)$, $f^{-1}(V)$ is a neighborhood of x, so it intersects A in some y. Then V intersects $f(A)$ at $f(y)$, so $f(x) \in \overline{f(A)}$. Hence $f(\overline{A}) \subseteq \overline{f(A)}$, so (1) implies (2).

(2) $\Rightarrow$ (3). Let B be closed in Y. Set $A = f^{-1}(B)$; we wish to prove that A is closed. Now, clearly $f(A) = f(f^{-1}(B)) \subseteq B$. Therefore, if $x \in \overline{A}$, $f(x) \in f(\overline{A}) \subseteq \overline{f(B)} \subseteq \overline{B} = B$, so that $x \in f^{-1}(B) = A$. Hence $\overline{A} \subseteq A$, implying that A is closed. In summary, (2) implies (3).

(3) $\Rightarrow$ (1) Let V be an open set of Y. Then, if $B = Y \setminus V$, $f^{-1}(B) = f^{-1}(Y) \setminus f^{-1}(V) = X \setminus f^{-1}(V)$. Yet $f^{-1}(B)$ is a closed set of Y by hypothesis, so $f^{-1}(V)$ is open, as desired. Hence (3) implies (1).

By proving that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1), we have shown that (1), (2), and (3) are equivalent. It suffices to prove (1) $\Leftrightarrow$ (4). We begin with (1) $\Rightarrow$ (4). Hence assume (1). Then take $x \in X$ and let V be a neighborhood of $f(x)$. $U = f^{-1}(V)$ is a neighborhood of x which definitionally satisfies $f(U) \subseteq V$, so (4) follows.

For (1) $\Rightarrow$ (4), let V be an open set of Y. Let x be a arbitrary point of $f^{-1}(V)$. Then $f(x) \in V$, so by hypothesis there is a neighborhood U_x of x such that $f(U_x) \subseteq V$ (that is, $U_x \subseteq f^{-1}(V)$). But $f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$ is the union of open sets and hence open. Therefore (4) implies (1). ☐

Definition 11 (Homeomorphisms)**.** Let $f : X \to Y$ be a continuous map of topological spaces. Then f is called a *homeomorphism* if there exists a continuous map $f^{-1} : Y \to X$ such that $f \circ f^{-1} = \mathrm{id}_Y$ and $f^{-1} \circ f = \mathrm{id}_X$. Equivalently, f is a homeomorphism if it is a bijection such that $U \subseteq X$ is open if and only if $f(U) \subseteq X$ is open.

Homeomorphisms are roughly the "isomorphisms" of topological spaces.

Theorem 6 (Constructing Continuous Functions)**.** *Let X, Y, and Z be topological spaces, and $f : X \to Y$ and $g : Y \to Z$ be maps of topological spaces. Then,*

1. *If $f : X \to Y$ is constant (maps all of X into a single point y_0 of Y), then f is continuous.*
2. *If A is a subspace of X, the inclusion function $j : A \to X$ is continuous.*
3. *If f and g are continuous, then $g \circ f$ is continuous.*
4. *If $f : X \to Y$ is continuous, and if A is a subspace of X, then the restriction $f|_A : A \to Y$ is continuous.*
5. *Let $f : X \to Y$ be continuous. If Z is a subspace of Y containing $f(X)$, then the induced map $f' : X \to Z$ is continuous. Similarly, if Y is a subspace of Z, then $f' : X \to Z$ is continuous.*
6. *Local Continuity: The map $f : X \to Y$ is continuous if X can be written as the union of open sets U_α such that $f|_{U_\alpha}$ is continuous for each α.*

Lemma 7 (The Gluing Lemma)**.** *Let $X = A \cup B$, where A and B are closed in X. If $f : A \to Y$ and $g : B \to Y$ are continuous, and $f|_{A\cap B} = g|_{A\cap B}$, then we may combine f and g to get a continuous map $h : X \to Y$ by setting $h(x) = f(x)$ if $x \in A$ and $h(x) = g(x)$ if $x \in B$.*

1.4 The Product and Subspace Topologies

Definition 12 (Product Topology)**.** Let X and Y be topological spaces. The *product topology* on $X \times Y$ is the topology having as basis the collection $\mathcal{B}$ of all sets of the form $U \times V$, where U is an open subset of X and V is an open subset of Y.

Theorem 8 (Product Basis)**.** *Suppose $\mathcal{B}$ is a basis for the topology of X and $\mathcal{C}$ is a basis for the topology of Y. Then the collection $\mathcal{D} = \{B \times C \mid B \in \mathcal{B}, C \in \mathcal{C}\}$ is a basis for the topology of $X \times Y$.*

Theorem 9 (Maps into Products)**.** *Let $f : A \to X \times Y$ be given by $f(a) = (f_1(a), f_2(a))$. Then f is continuous if and only if $f_1 : A \to X$ and $f_2 : A \to Y$, called the coordinate functions of f, are continuous.*

Definition 13 (Standard Topology on $\mathbb{R}^n$)**.** The *standard topology*, or *product topology*, on $\mathbb{R}^n$ is given by the product topology of n copies of $\mathbb{R}$ with the standard topology.

Theorem 10. *Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be the projections given by $(x, y) \mapsto x$ and $(x, y) \mapsto y$. Then the collection $\mathscr{S} = \{\pi_1^{-1}(U) \mid U \text{ open in } X\} \cup \{\pi_2^{-1}(V) \mid V \text{ open in } X\}$ is a subbasis for $X \times Y$.*

Definition 14 (Subspace Topology)**.** Let X be a topological space with topology $\mathcal{T}$. Take a subset Y. Then the *subset topology on Y* (also called the *induced topology on Y*) is the collection

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}.$$

We call $(Y, \mathcal{T}_Y)$ a *subspace* of $(X, \mathcal{T})$.

Lemma 11 (Subspace Basis)**.** *If $\mathcal{B}$ is a basis for the topology of X then the collection $\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$ is a basis for the subspace topology on Y.*

Theorem 12 (Condition for Being Closed in a Subspace)**.** *Let Y be a subspace of X. Then a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y.*

Corollary 12.1. *Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.*

Theorem 13 (Closures in Subspaces)**.** *Let Y be a subspace of X with subset A. Then, if $\overline{A}$ denotes the closure of A in X, the closure of A in Y equals $\overline{A} \cap Y$.*

1.5 The Quotient Topology

Definition 15 (Quotient Map)**.** Let X and Y be topological spaces with a surjective map $\pi : X \to Y$. Then, if a subset $U \subseteq Y$ is open if and only if $p^{-1}(U) \subseteq X$ is open, the map π is said to be a *quotient map.*

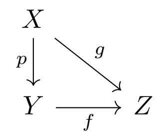
Definition 16 (Open and Closed Map)**.** A map $f : X \to Y$ of topological spaces is called *open* if for each open set U of X, the set $f(U)$ is open in Y. Similarly, f is called *closed* if for each closed set A of X, then set $f(A)$ is closed in Y.

Theorem 14 (Examples of Quotient Maps)**.** *Let $f : X \to Y$ be a surjective continuous map of topological spaces. Then if $f : X \to Y$ is open or closed, it is a quotient map.*

Definition 17 (Quotient Topology)**.** If X is a space, A is a set, and $\pi : X \to A$ is a surjective map, there exists exactly one topology $\mathcal{T}$ on A relative to which p is a quotient map, called the *quotient topology induced by π*. Explicitly, this topology is defined as the collection $\mathcal{T} = \{U \subseteq A \mid p^{-1}(U) \text{ is open in } X\}$.

Definition 18 (Quotient Space)**.** Let X be a topological space, and let X^* be a partition of X into disjoint subsets whose union is X. Then let $\pi : X \to X^*$ be the surjective map that carries each point of X to the element of X^* containing it. Then in the quotient topology induced by π, the space X^* is called a *quotient space* of X.

Theorem 15 (Universal Property)**.** *Let $p : X \to Y$ be a quotient map. Let Z be a space and $g : X \to Z$ be a map that is constant on each set $p^{-1}(\{y\})$ for $y \in Y$. Then g induces a unique map $f : Y \to Z$ such that $f \circ p = g$. The induced map f is continuous (resp. a quotient map) if and only if g is continuous (resp. a quotient map). Graphically, we have the following commutative diagram:*



Corollary 15.1. *Let $g : X \to Z$ be a surjective continuous map. Define the following partition*

$$X^* = \{g^{-1}(\{z\}) \mid z \in Z\}.$$

Then, if X^* is given the quotient topology, the map g induces a bijective continuous map $f : X^* \to Z$, which is a homeomorphism if and only if g is a quotient map. Also, if Z is Hausdorff, so is X^*.

1.6 Metric Spaces

Definition 19 (Metric)**.** A *metric* on a set X is a function $d : X \times X \to \mathbb{R}$ such that

1. $d(x, y) \geq 0$ for all $x, y \in X$; equality holds if and only if $x = y$ (positive definite).
2. $d(x, y) = d(y, x)$ for all $x, y \in X$ (symmetric).
3. $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$ (triangle inequality).

Definition 20 (ϵ-ball)**.** Let X be a set with metric d. Then, given $\epsilon > 0$, the set $B_d(x, \epsilon) = \{y \mid d(x, y) < \epsilon\}$ is called *the ϵ-ball centered at x*. When the metric d is clear, we may also denote the ε-ball by $B(x, \epsilon)$.

Definition 21 (Topology from a Metric)**.** Let X be a set with metric d. Then the collection of all ϵ-balls $B_d(x, \epsilon)$ (for any $x \in X$) is a basis for a topology on X, called the *metric topology induced by d.*

Definition 22 (Metrizable Topology)**.** If X is a topological space, X is said to be metrizable if there exists a metric d on X that induces the topology of X. A *metric space* is a metrizable space X together with a specific metric d that gives the topology of X.

Definition 23 (Boundedness)**.** Let X be a metric space with metric d. A subset A of X is said to be *bounded* if there is some number M such that $d(a_1, a_2) \leq M$ for every pair $a_1, a_2 \in A$. If A is furthermore nonempty, we define the *diameter* of A to be the number $\operatorname{diam} A = \sup\{d(a_1, a_2) \mid a_1, a_2 \in A\}$.

Note that many metrics induce the same topology. For example, if d is a metric, then the metric $d'(x, y) = 2d(x, y)$ is also a metric which induces the same topology. Therefore, the diameter of a set is not a topological property. What might be more surprising, however, is that boundedness is not a topological property either. Consider the following metric.

Theorem 16 (Standard Bounded Metric)**.** *Let X be a metric space with metric d. Define $\overline{d} : X \times X \to \mathbb{R}$ by*

$$\overline{d}(x, y) = \min\{d(x, y), 1\}.$$

Then $\overline{d}$ is a metric, called the standard bounded metric *that induces the same topology as d.*

Proof. The fact that $\overline{d}$ is a metric is trivial. However, proving that $\overline{d}$ generates the same topology as d requires more creativity. Let $\mathcal{B}$ be the collection of all ϵ-balls under the metric d, and let $\mathcal{B}'$ be the collection of all ϵ-balls with $\epsilon < 1$ (here, the metrics d and $\overline{d}$ coincide). Clearly, $\mathcal{B} \supseteq \mathcal{B}'$, so the metric topology generated by $\mathcal{B}$ is at least as fine as the topology generated by $\mathcal{B}'$. On the other hand, for each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ satisfying $x \in B' \subseteq B$ (simply take a small enough ball). Therefore, by Lemma 2, the topology generated by $\mathcal{B}'$ is finer than the topology generated by $\mathcal{B}$. Hence they are equal, as desired. $\square$

This leads to the following generalization of the above fact:

Lemma 17 (Comparing Metric Spaces)**.** *Let d and d' be two metrics on the set X. Then let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies they induce, respectively. Then $\mathcal{T}'$ is finer than $\mathcal{T}$ if and only if for each $x \in X$ and each $\epsilon > 0$, there exists $\delta > 0$ such that*

$$B_{d'}(x, \delta)$$

Proof. By the same strategy as Theorem 16, namely using Lemma 2. □

Definition 24 (Common Examples of Metric Spaces)**.** The following are all examples of metric spaces:

1. Given $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$, we define the *norm* of x by the equation $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$ and the *Euclidean metric* d on $\mathbb{R}^n$ by the equation $d(x, y) = \|x - y\|$.
2. The *square metric* ρ is defined by the equation $\rho(x, y) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}$.

Note that proving the former is actually a metric is nontrivial.

Theorem 18 (Comparing Topologies on $\mathbb{R}^n$)**.** *The topologies on $\mathbb{R}^n$ induced by the Euclidean metric d and the square metric ρ are the same as the product topology on $\mathbb{R}^n$.*

Proof. Proving that the Euclidean metric and square metric is trivial using Lemma 17. On the other hand, to prove the product topology and square metric topology are the same, first notice that any basis element of the square metric topology is a basis element of the product topology, so the product topology is finer. Next, consider an arbitrary basis element for the product topology $B = (a_1, b_1) \times \cdots \times (a_n, b_n)$. Then, for each $x \in B$ and each i, there exists ϵ_i such that $(x_i - \epsilon) \subseteq (a_i, b_i)$. Then $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_n\}$ satisfies $B_\rho(x, \epsilon) \subseteq B$, so Lemma 2 is applicable and the square metric topology is finer. Hence they are equal. □

Theorem 19 (Continuity in Metric Spaces)**.** *Let $f : X \to Y$ be metric spaces with metrics d_X and d_Y, respectively. Then f is continuous if and only if, given $x \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \epsilon.$$

Lemma 20 (The Sequence Lemma)**.** *Let X be a topological space and let $A \subseteq X$. If there is a sequence of points of A converging to x, then $x \in \overline{A}$. The converse holds if X is metrizable.*

Theorem 21 (Continuity by Sequences)**.** *Let $f : X \to Y$ be a map of topological spaces. If f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is metrizable.*

Theorem 22 (Continuous Maps in $\mathbb{R}^n$)**.** *The addition, subtraction, and multiplication operations are continuous maps $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and the quotient operation is a continuous map $\mathbb{R} \times (\mathbb{R} \setminus \{0\}) \to \mathbb{R}$.*

If X is a topological space, and if $f, g : X \to \mathbb{R}$ are continuous, then $f + g$, $f - g$, and $f \cdot g$ are continuous. If $g(x) \neq 0$ for all x, then f/g is continuous. Therefore, for example, all polynomial functions are continuous.

Definition 25 (Uniform Convergence)**.** Let $f_n : X \to Y$ be a sequence of functions from a set X to a metric space Y. Then (f_n) *converges uniformly* to a function $f : X \to Y$ if, given $\epsilon > 0$, there exists an integer N such that $d(f_n(x), f(x)) < \epsilon$ for all $n > N$ and all $x \in X$.

Compare this definition to pointwise convergence, which given by the following:

Definition 26 (Pointwise Convergence)**.** Let $f_n : X \to Y$ be a sequence of functions from a set X to a metric space Y. Then (f_n) *converges pointwise* to a function $f : X \to Y$ if, for each point $x \in X$, the sequence $(f_n(x))$ converges to $f(x)$.

Why is uniform convergence significant compared to pointwise convergence? In short, uniform convergence preserves properties "better" than pointwise convergence. A simple example: continuity.

Example 1. Consider the function $f_n : [0,1] \to [0,1]$ given by $f_n(x) = x^n$. Each f_n is continuous by Theorem 22, but (f_n) converges pointwise to the function

$$f(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$$

which is discontinuous. Therefore, pointwise convergence does not preserve continuity.

In comparison, uniform convergence does preserve continuity. Formally,

Theorem 23 (Uniform Limit Theorem)**.** *Let $f_n : X \to Y$ be a sequence of continuous functions from the topological space X to the metric space Y. If (f_n) converges uniformly to f, then f is continuous.*

Proof. Let $V \subseteq Y$ be open and let x_0 be a point of $f^{-1}(V)$. By Theorem 5, it suffices to find a neighborhood U of x_0 such that $f(U) \subseteq V$. For this, let $y_0 = f(x_0)$. Now, we may choose ϵ so $B(y_0, \epsilon)$ is contained in V. Then, using uniform convergence, choose N so that for all $n \geq N$ all $x \in X$, $d(f_n(x), f(x)) < \epsilon/3$.

Using continuity of f_N, there is a neighborhood U of x_0 such that f_N carries U into the $\epsilon/3$-ball centered at $f_N(x_0)$. Then, if $x \in U$,

$$\begin{aligned} d(f(x), f_N(x)) &< \epsilon/3 \text{ by choice of } N. \\ d(f_N(x), f_N(x_0)) &< \epsilon/3 \text{ by choice of } U. \\ d(f_N(x_0), f(x_0)) &< \epsilon/3 \text{ by choice of } N. \end{aligned}$$

Hence, by using the triangle inequality, we see that $d(f(x)), f(x_0)) < \varepsilon$, as desired. Therefore, f carries U into $B(y_0, \epsilon)$ and hence into V, as desired. □

Definition 27 (Isometry)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. Then $f : X \to Y$ is an *isometry* if for any $a, b \in X$ one has $d_X(a, b) = d_Y(f(a), f(b))$. Now, f is automatically injective, but if it is furthermore bijective, then it is called an *isometric isomorphism.*

2 Connectedness

2.1 Connectedness

Definition 28 (Connected and Disconnected Spaces)**.** Let X be a topological space. A *separation* of X is a pair U, V of disjoint nonempty open subsets of X whose union is X. A space X with a separation is *disconnected*, and a space with no separation is *connected.*

Equivalently, a space is connected if and only if the only subsets of X that are clopen (both open and closed) in X are the empty set and X itself.

Lemma 24. *Suppose Y is a subspace of X. Then a separation of Y is a pair of disjoint nonempty sets A and B whose union is Y, neither of which contains a limit of the other.*

Proof. Suppose that A and B form a separation of Y. Then A is open and closed in Y. Now, the closure of A in Y is equal to $\overline{A} \cap Y$ (where $\overline{A}$ is the closure of A in X). Since A is closed in Y, however, $A = \overline{A} \cap Y$. This implies that $\overline{A} \cap B = \varnothing$, and since $\overline{A}$ is the union of A and its limit points, B cannot contain any limit points of A. A similar argument shows that A contains no limit point of B.

Suppose that A and B are disjoint nonempty sets whose union is Y, neither of which contains a limit point of the other. Yet then $\overline{A} \cap B = \varnothing$ and $A \cap \overline{B} = \varnothing$, whence $\overline{A} \cap Y = A$ and $\overline{B} \cap Y = B$. Therefore, A and B are closed in Y. Since they are complementary, this implies they are both open and form a separation. □

Lemma 25. *If A, B is a separation of X, and Y is a connected subspace of X, then $Y \subseteq A$ or $Y \subseteq B$.*

Theorem 26 (Building Connected Spaces)**.** *Let X be a topological space.*

1. *If $\{A_\alpha\}$ is a collection of connected subspaces of X with a point p in common, then $\bigcup A_\alpha$ is connected.*

2. *Let A be a connected subspace of X. If $A \subseteq B \subseteq \overline{A}$, then B is a connected subset of X.*

3. *If X is connected and $f : X \to Y$ is a continuous map, then $\operatorname{im} f \subseteq Y$ is connected.*

4. *If X and Y are connected topological spaces, $X \times Y$ is connected. By induction, this implies immediately that all finite products of connected spaces are connected.*

Proof. For instruction, we will prove all four of these results.

(1). Let $Y = \bigcup A_\alpha$, and assume it has the separation C, D. Now, p is either in C or D. Assume without loss of generality it is contained in C. But then, for each α, since A_α is connected, Lemma 25 and the fact that A_α contains p implies that $A_\alpha \subseteq C$. But then $Y \subseteq C$, implying that D is empty, a contradiction.

(2). Suppose $B = C \cup D$ is a separation of B. Again by Lemma 25, A lies in either C or D. Assume without loss of generality that $A \subseteq C$, so that $\overline{A} \subseteq \overline{C}$. But $\overline{C}$ and D are disjoint (see Lemma 24), so $B \subseteq \overline{A} \subseteq C$, a contradiction with the fact that D is nonempty.

(3). Clearly, by restriction onto the image, it suffices to consider surjective continuous maps $f : X \to Y$. Now, if Y has separation A, B, then $f^{-1}(A), f^{-1}(B)$ is a separation of X (note that by surjectivity both preimages are nonempty). Hence if Y is disconnected, X is disconnected, as desired.

(4). Choose a distinguished "base point" $(a, b) \in X \times Y$. Now, the "horizontal slice" $X \times \{b\}$ is homeomorphic to X and hence connected. Similarly, $\{x\} \times Y$ is homeomorphic to Y and hence connected. Therefore,

$$T_x = (X \times b) \cup (x \times Y),$$

as the union of connected spaces with the point (x, b) in common, is connected. But notice that each T_x contains the point (a, b), so the union $X \times Y = \bigcup_{x \in X} T_x$ is also connected, as desired. □

2.2 The Order Topology and Connected Subsets of $\mathbb{R}$

Next, we will discuss connected subspaces of the real line. For this, we define the *order topology*.

Definition 29 (Intervals)**.** Let X be a totally ordered set and take elements a and b of X such that $a < b$. Then, there are four subsets of X called the *intervals* determined by a and b:

1. $(a, b) = \{x \mid a < x < b\}$ (*open interval*).

2. $(a, b] = \{x \mid a < x \leq b\}$ (*half-open interval*).

3. $[a, b) = \{x \mid a \leq x < b\}$ (*half-open interval*).

4. $[a, b] = \{x \mid a \leq x \leq b\}$ (*closed interval*).

Definition 30 (Order Topology)**.** Let X be a totally ordered set with more than one element. Then let $\mathcal{B}$ be collection of all open intervals of X, as well as

1. all intervals of the form $[a_0, b)$, where a_0 is the smallest element (if there is any) of X,

2. all intervals of the form $(a, b_0]$, where b_0 is the largest element (if there is any) of X.

Then the collection $\mathcal{B}$ is a basis for a topology on X, which is called the *order topology*.

Definition 31 (Open and Closed Rays)**.** Let X be a totally ordered set and choose $a \in X$. Then, there are four subsets of X called the *rays* determined by a:

1. $(a, +\infty) = \{x \mid x > a\}$ (*open ray*).

2. $(-\infty, a) = \{x \mid x < a\}$ (*open ray*).

3. $[a, +\infty) = \{x \mid x \geq a\}$ (*closed ray*).

4. $(-\infty, a] = \{x \mid x \leq a\}$ (*closed ray*).

It is easy to prove that the open rays are open and the closed rays are closed in the order topology on X.

Definition 32 (Linear Continuum)**.** A totally ordered set L with more than one element is called a *linear continuum* if (1) L has the least upper bound property (that is, any non-empty subset of L has a least upper bound) and (2) if $x < y$, there exists $z \in L$ such that $x < z < y$.

Theorem 27 (Linear Continua are Connected)**.** *If L is a linear continuum in the order topology, then L is connected, and so are intervals and rays in L.*

Proof. Call a subspace Y of L *convex* if for every pair of points $a, b \in Y$ with $a < b$, the interval $[a, b] \subseteq L$. We will show that all convex subspaces Y of L are connected; since L, intervals, and rays are convex, this suffices to show the desired result.

Assume that Y is the union of disjoint nonempty open sets A and B. Choose $a \in A$ and $b \in B$. By relabeling, we may force $a < b$. Now, the interval $[a, b]$ of points of L is contained in Y. Hence $[a, b]$ is the union of the disjoint sets $A_0 = A \cap [a, b]$ and $B_0 = B \cap [a, b]$. Both A_0 and B_0 are open in $[a, b]$ and nonempty because $a \in A_0$ and $b \in B_0$. Thus A_0 and B_0 are a separation of $[a, b]$.

Now let $c = \sup A_0$. We show that c belongs neither to A_0 nor to B_0, which contradicts the fact that $[a, b]$ is the union of A_0 and B_0. Both cases are proven below:

Case 1: Suppose that $c \in A_0$. Then $c \neq b$, so either $c = a$ or $a < c < b$. Because A_0 is open in $[a, b]$, there must be some interval of the form $[c, e)$ contained in A_0. But then because of the order property of the linear continuum, we can choose z of L such that $c < z < e$. Then $z \in A_0$, contrary to the fact that c is an upper bound for A_0.

Case 2: Suppose that $c \in B_0$. Then $c \neq a$, so either $c = b$ or $a < c < b$. In either case, it follows from the fact that B_0 is open in $[a, b]$ that there is some interval of the form $(d, c]$ contained in B_0. If $c = b$, we have a contradiction, for d is a smaller upper bound on A_0 than c. On the other hand, if $c < b$, we note that $(c, b]$ does not intersect A_0 because c is an upper bound on A_0. Then $(d, b] = (d, c] \cup (c, b]$ also does not intersect A_0, so d is a smaller upper bound on A_0 than c, a contradiction. □

Corollary 27.1. *Since $\mathbb{R}$ is a linear continuum, $\mathbb{R}$ is connected, and so are intervals and rays in $\mathbb{R}$.*

From this corollary follows the intermediate value theorem, which we state here in greater generality than is usually covered in calculus.

Theorem 28. *Let $f : X \to Y$ be a continuous map between a connected space X and an ordered set Y with the order topology. If a and b are two points of X and r is a point of Y lying between $f(a)$ and $f(b)$, then there exists a point c of X such that $f(c) = r$.*

Proof. Assume the hypotheses of the theorem. Then $A = f(X) \cap (-\infty, r)$ and $B = f(X) \cap (r, +\infty)$ are disjoint and nonempty, since one contains $f(a)$ and the other contains $f(b)$. Furthermore, as the intersection of an open ray in Y with the open set $f(X)$, both A and B are open in $f(X)$. But $f(X)$, as the image of a connected set, is connected, so we cannot have $A \cup B = f(X)$. Hence there must a point contained in $f(X) \setminus (A \cup B)$, and by construction of A and B the only possibility for such a point is r, as desired. □

2.3 Path-Connectedness

Definition 33 (Path and Path-Connected)**.** Given points x and y of the space X, a *path in X* from x to y is a continuous map $f : [a, b] \to X$ of some closed interval in the real line into X such that $f(a) = x$ and $f(b) = y$. X is said to be *path-connected* if every pair of points in X can be joined by a path in X.

Lemma 29. *A path-connected space is connected.*

Proof. Suppose that X is not connected; i.e. that there exists a separation A, B of X. Let $f : [a, b] \to X$ be any continuous map (i.e. any path). Then, as the continuous image of a connected set, the image $f([a, b])$ is connected, so by Lemma 25 it is contained in either A or B. But then there is no path from any point $x \in A$ to a point $y \in B$, so X is not path connected. Contraposition shows the desired result. □

2.4 Components and Local Connectedness

Definition 34 (Connected Components)**.** Given X, define an equivalence relation on X by setting $x \sim y$ if there is a connected subspace of X containing x and y. Then the equivalence classes are called the *components* or *connected components* of X.

Theorem 30 (Equivalent Notion of Components)**.** *The components of X are connected disjoint subspaces of X whose union is X such that each nonempty connected subspace of X intersects only one of them.*

Definition 35 (Path Components)**.** Given X, define an equivalence relation on X by setting $x \sim y$ if there is a path in X from x to y. The equivalence classes are called the *path components of X*.

Theorem 31 (Equivalent Notion of Path Components)**.** *The path components of X are path-connected disjoint subspaces of X whose union is X such that each nonempty path-connected subspace of X intersects only one of them.*

Definition 36 (Locally Connected and Locally Path Connected)**.** A space X is *locally connected at x* if for every neighborhood U of x, there is a connected neighborhood V of x contained in U. If X is locally connected at each of its points, it is called *locally connected*. Similarly, X is *locally path connected at x* if for every neighborhood U of x, there is a path connected neighborhood V of x contained in U, and X is *locally path connected* if it is locally path connected at each of its points.

Theorem 32 (Locally Connected Criterion)**.** *A space X is locally connected if and only if for every open set U of X, each component of U is open in X.*

Proof. Suppose X is locally connected and $U \subseteq X$ is open. Let C be a component of U. If x is a point of C, we can choose a connected neighborhood V_x of x such that $V_x \subseteq U$. Since V_x is connected, it must lie entirely in the component C of U. Therefore, $C = \bigcup_{x \in C} V_x$ is open.

Conversely, suppose that components of open sets in X are open. Given a point x of X and a neighborhood U of x. Let C be the component of U containing x. Now C is connected; since it is open in X by hypothesis, X is locally connected at x. □

Theorem 33 (Locally Path Connected Criterion)**.** *A space X is locally path connected if and only if for every open set U of X, each path component of U is open in X.*

Theorem 34. *If X is a topological space, each path component of X lies in a component of X. If X is locally path connected, then the components and the path components of X are the same.*

Proof. Let C be a component of X, choose $x \in C$. Let P be the path component of X containing x. Plainly, since P is connected, $P \subseteq C$. Therefore, it suffices to show that if X is locally path connected, $P = C$. Suppose that $P \subsetneq C$. Let Q denote the union of all the path components of X that are different from P and intersect C. Each of them necessarily lies in C, so that

$$C = P \cup Q.$$

Because X is locally path component of X is open in X. Therefore, P (which is a path component) and Q (which is a union of path components) are open in X, so they are a separation of C. But this is a contradiction with the fact that C is connected, so our assumption $P \subsetneq C$ was false. □

3 Seperation and Countability Axioms

3.1 T_1 Spaces

Definition 37 (Limit Points)**.** Suppose that X is a topological space with subset A. Then x is a *limit point* of A if every neighborhood of A intersects A in some point other than x itself.

Theorem 35 (Limit Points and Closures)**.** *Let A be a subset of the topological space X and let A' be the set of all the limit points of A. Then $\overline{A} = A \cup A'$.*

Proof. Suppose $x \in A'$. Then every neighborhood of x intersects A in a point different from x. Therefore, by Theorem 4, $x \in \overline{A}$. Since $A \subseteq \overline{A}$, $A \cup A' \subseteq \overline{A}$. To prove the other inclusion, let $x \in \overline{A}$ be arbitrary – we seek to show that $x \in A \cup A'$. If $x \in A$, the result follows. Otherwise, if $x \notin A$ recall that Theorem 4 states that any neighborhood of x intersects A. Since $x \notin A$ by assumption, this means any neighborhood of x intersects A in a point other than x, and hence $x \in A'$, as desired. □

Corollary 35.1. *A subset of a topological space is closed if and only if it contains all its limit points.*

Definition 38 (T_1 Spaces)**.** Let X be a topological space. Then the T_1 *axiom* states that for each pair x_1, x_2 of distinct points of X, there exist neighborhoods U_1 and U_2 of x_1 and x_2 such that U_2 does not contain x_1 and U_1 does not contain x_2. A topological space X is called a T_1 *space* if it satisfies the T_1 axiom.

Theorem 36 (Alternative Characterization of T_1 Spaces)**.** *Let X be a topological space. Then X is a T_1 space if and only if every finite subset of X is closed.*

Proof. Notice that every finite subset of X is closed if and only if every singleton of X is closed. Now, if every singleton of X is closed, then $X \setminus \{x_2\}$ and $X \setminus \{x_1\}$ are both open – in particular, $X \setminus \{x_2\}$ is a neighborhood of x_1 not containing x_2, and $X \setminus \{x_1\}$ is a neighborhood of x_2 not containing x_1. Hence X is T_1.

On the other hand, suppose X is a T_1 space. Let $x \in X$ and consider the closure of $\{x\}$. Suppose that $x_0 \neq x$ is an arbitrary point. Then there exists a neighorhood U of x_0 not containing x. Then $X \setminus U$ is a closed set containing $\{x\}$ but not x_0, so x_0 is not in the closure of $\{x\}$. This proves that the closure of $\{x\}$ is simply $\{x\}$, which implies $\{x\}$ is closed, as desired. □

Theorem 37 (Limit Points in T_1 Spaces)**.** *Let X be a T_1 space with subset A. Then the point x is a limit point of A if and only if every neighborhood of x contains infinitely many points of A.*

Proof. Clearly, if every neighborhood of x intersects A in infinitely many points, it certainly intersects A in some point other than x itself, so that x is a limit point of A.

Conversely, suppose that x is a limit point of A, and suppose for the sake of contradiction some neighborhood U of x intersects A in only finitely many points. Then U also intersects $A \setminus \{x\}$ in finitely many points $\{x_1, \ldots, x_m\}$. Then, the set $X \setminus \{x_1, \ldots, x_m\}$ is an open set of X since $\{x_1, \ldots, x_m\}$ is closed by Theorem 36. But then $U \cap (X \setminus \{x_1, \ldots, x_m\})$ is a neighborhood of x disjoint from $A \setminus \{x\}$. This implies that x is not a limit point of A, a contradiction. □

3.2 Hausdorff Spaces

Definition 39 (Hausdorff Spaces)**.** Let X be a topological space. Then the T_2 *axiom* states that for each pair x_1, x_2 of distinct points of X, there exist disjoint neighborhoods U_1 and U_2 containing x_1 and x_2 respectively. In other words, the T_2 axiom guarantees that we have a strong topological method to distinguish any two points. A topological space X is called a *Hausdorff space* if it satisfies the T_2 axiom.

Definition 40 (Convergent Sequences)**.** Let X be a topological space. Then a sequence $x_1, x_2, \ldots$ of points of X *converges* to $x \in X$ if, for any neighborhood U of x, there exists a positive integer N such that $x_n \in U$ for all $n \geq N$.

Theorem 38 (Properties of Hausdorff Spaces)**.** *Let X be a Hausdorff space. Then,*

1. *X is a T_1 space; every finite point set in a Hausdorff space X is closed.*
2. *A sequence of points of X converges to at most one point of X.*
3. *The product of two Hausdorff spaces is a Hausdorff space.*
4. *A subspace of a Hausdorff space is a Hausdorff space.*

Proof. Of these, we will only prove (2). For this, suppose that x_n is a sequence of points of X that converges to x. Then, if $y \neq x$, let U and V be disjoint neighborhoods of x and y, respectively. Since U contains x_n for all but finitely many values of n, the set V cannot. Therefore, x_n cannot converge to y. □

Note that there are T_1 spaces where sequences converge to multiple points. A natural example, called *the Zariski topology on k^n*, comes from algebraic geometry and commutative algebra.

Definition 41 (Zariski Topology on k^n)**.** Let k be a field and $k^n = \mathbb{A}_k^n$ be n-dimensional affine space over k. Then the *Zariski topology on k^n* is given by defining a set in k^n to be closed if it is the zero locus of a nonempty collection of polynomials in $k[x_1, \ldots, x_n]$.

The details of the Zariski topology are quite interesting but left to other notes. However, it is worth mentioning that the zero locus of any collection S of polynomials in $k[x_1, \ldots, x_n]$ is equal to the zero locus of the ideal generated by S (this is useful in proving that the Zariski topology is indeed a topology, as taking unions and intersections can be interpreted as taking products and sums of ideals). Furthermore, with Hilbert's Basis Theorem, which states that a polynomial ring over a Noetherian ring is Noetherian, we may recall that any ideal of $k[x_1, \ldots, x_n]$ is generated by finitely many polynomials, and therefore "collection of polynomials" can be replaced by "finite collection of polynomials" without issue.

Example 2. The Zariski topology on $\mathbb{Q}^n$ is a T_1 space, but not a Hausdorff space. Furthermore, there exists a sequence which converges to every point.

Proof. The Zariski topology on $\mathbb{Q}^n$ is T_1 because individual points are closed – the zero locus of the collection $\{(x_1 - k_1), \ldots, (x_n - k_n)\}$ is precisely $\{(k_1, \ldots, k_n)\}$, proving that the latter is closed. However, it is not Hausdorff because any two nonempty open sets intersect, so we cannot find disjoint neighborhoods.

Now we construct a sequence which converges to every point. It is well-known that there are countably many nonzero n-dimensional polynomials over $\mathbb{Q}^n$. Therefore, we may number them $p_1, p_2, \ldots$. Next, choose $x_k \in \mathbb{Q}^n$ to be such that $p_j(x_k) \neq 0$ for all $j < k$. To see why this is possible, define the polynomial $P_k = \prod_{j<k} p_j$. Since P_j is the product of nonzero polynomials, it is nonzero, and therefore there exists a point x_k for which $P_k(x_k)$ is nonzero. Yet if $P_k(x_k)$ is nonzero, $p_j(x_k)$ is nonzero for each $j < k$.

Now, I claim that (x_n) converges to every point in $\mathbb{Q}^n$ in the Zariski topology. To see why, let $x \in \mathbb{Q}^n$ be an arbitrary point and U be an arbitrary neighborhood of x. Then $\mathbb{Q}^n \setminus U$ is a closed set, so it is the zero locus of a nonempty collection S of polynomials in $k[x_1, \ldots, x_n]$. Pick any polynomial $p_N \in S$. Then, for all $n > N$, $p_N(x_n) \neq 0$, so x_n is not in the zero locus of S. But this means precisely that x_n is not contained in $\mathbb{Q}^n \setminus U$, so it is contained in U. Therefore (x_n) converges to x, as desired. □

3.3 The Countability Axioms

Definition 42 (Countable Base)**.** A space X has a *countable basis at x* if there is a countable colection $\mathcal{B}$ of neighborhoods of x such that each neighborhood of x contains at least one of the elements of $\mathcal{B}$. A space that has a countable basis at each point is said to satisfy the *first countability axiom* or be *first-countable.*

For example, all metric spaces satisfy the first countability axiom.

Theorem 39 (Consequences of First-Countability)**.** *Let X be a topological space. Then*

1. *Let A be a subset of X. If there is a sequence of points of A converging to x, then $x \in \overline{A}$; the converse holds if X is first-countable.*

2. *Let $f : X \to Y$. If f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is first-countable.*

Definition 43 (Second Countability Axiom)**.** If a space X has a countable basis for its topology, then X is said to *satisfy the second countability axiom* or be *second-countable.*

Theorem 40 (Countability Axiom Preservation)**.** *A subspace of a first-countable space is first-countable, and a countable product of first-countable spaces is first-countable. A subspace of a second-countable space is second-countable, and a countable product of second-countable spaces is second-countable.*

Definition 44 (Dense)**.** A subset A of a space X is said to be *dense* in X if $\overline{A} = X$.

Theorem 41. *Suppose that X has a countable basis $\{B_n\}$. Then:*

1. *Every open covering of X contains a countable subcollection covering X.*

2. *There exists a countable subset of X that is dense in X.*

Proof.
1. Let $\mathcal{A}$ be an open covering of X. For each positive integer n for which it is possible, choose an element A_n of $\mathcal{A}$ containing the basis element B_n. The collection $\mathcal{A}'$ of the sets A_n is countable, since it is indexed with a subset J of the positive integers. Furthermore, it covers X: given a point $x \in X$, we can choose an element A of $\mathcal{A}$ containing x. Since A is open, there is a basis element B_n such that $x \in B_n \subseteq A$. Because B_n lies in an element of $\mathcal{A}$, the index n belongs to the set J, so A_n is defined: since A_n containing B_n, it contains x. Thus $\mathcal{A}'$ is countable subcollection of $\mathcal{A}$ that covers X.

2. Choose a point x_n from each nonempty basis element B_n. Let D be the set consisting of all the x_n. Then D is dense in X as any point x of X, every basis element containing x intersects D, so x belongs to $\overline{D}$. □

Definition 45 (Seperable)**.** Let X be a topological space. Then X is called *separable* if there is a countable dense subset of X. Notice that any second-countable space is separable.

3.4 The Seperation Axioms

Definition 46 (Regular and Normal)**.** Suppose that one-point sets are closed in X.

1. Then X is said to be *regular* if for each pair consisting of a point x and a closed set B disjoint from x, there exist disjoint open sets containing x and B, respectively.

2. The space X is said to be *normal* if for each pair A, B of disjoint closed sets of X, there exist disjoint open sets containing A and B, respectively.

It is clear that a regular space is Hausdorff and a normal space is regular.

Lemma 42. *Let X be a topological space. Let one-point sets in X be closed.*

1. *X is regular if and only if given a point $x \in X$ and a neighborhood U of x, there is a neighborhood V of x such that $\overline{V} \subseteq U$.*

2. *X is normal if and only if given a closed set A and an open set U containing A, there is an open set V containing A such that $\overline{V} \subseteq U$.*

Proof. We will only do (1), the proof for (2) is the same (replacing x with A). Suppose that X is regular, and suppose that the point x and the neighborhood U of x are given. Let $B = X \setminus U$; then B is a closed set. By hypothesis, there exist disjoint open sets V and W containing x and B, respectively. The set $\overline{V}$ is disjoint from B, since if $y \in B$, the set W is a neighborhood of y disjoint from V. Therefore, $\overline{V} \subseteq U$.

To prove the converse, suppose the point x and the closed set B not containing x are given. Let $U = X \setminus B$. By hypothesis, there is a neighborhood V of x such that $\overline{V} \subseteq U$. The open sets V and $X \setminus \overline{V}$ are disjoint open sets containing x and B respectively. Thus X is regular. □

Theorem 43 (Subspace/Product Separation Axioms)**.** *A subspace of a Hausdorff space is Hausdorff; a product of Hausdorff spaces is Hausdorff. Similarly, a subspace of a regular space is regular; a product of regular spaces is regular.*

Theorem 44. *Every regular space with a countable basis is normal.*

Proof. Let X be a regular space with a countable basis $\mathcal{B}$. Let A and B be disjoint closed subsets of X. Each point x of A has a neighborhood U not intersecting B (for example, $X \setminus B$). Using regularity, choose a neighborhood V of x whose closure lies in U; finally, choose an element of $\mathcal{B}$ containing x and contained in V. By choosing such a basis element for each x in A, we construct a covering of A by open sets whose closures do not intersect B. Then, by Theorem 41, there is a countable subcovering of A which we call $\{U_n\}$. Similarly, choose a countable collection $\{V_n\}$ of open sets covering B such that each set $\overline{V}_n$ is disjoint from A.

The sets $U = \bigcup U_n$ and $V = \bigcup V_n$ are open sets containing A and B, but they are not necessarily disjoint. Therefore, we need to make them disjoint using the following trick:

$$U'_n = U_n \setminus \bigcup_{i=1}^{n} \overline{V}_i \text{ and } V'_n = V_n \setminus \bigcup_{i=1}^{n} \overline{U}_i.$$

Each set U'_n is open, begin the difference of an open set U_n and a closed set $\bigcup_{i=1}^{n} \overline{V}_i$. Similarly, each set V'_n is open. The collection $\{U'_n\}$ covers A, because each x in A belongs to U_n for some n and x belongs to *none* of the sets $\overline{V}_i$. Similarly, the collection $\{V'_n\}$ covers B. Therefore, $U' = \bigcup U'_n$ and $V' = \bigcup V'_n$, which are clearly disjoint, are the necessary disjoint open sets containing A and B. □

Theorem 45. *Every metrizable space is normal.*

Proof. Let X be a metrizable space with metric d. Let A and B be disjoint closed subsets of X. For each $a \in A$, choose ϵ_a so that the ball $B(a, \epsilon_a)$ does not intersect B. Similarly, for each $b \in B$, choose ϵ_b so that the ball $B(b, \epsilon_b)$ does not intersect A. Then define

$$U = \bigcup_{a \in A} B(a, \epsilon_a/2) \text{ and } V = \bigcup_{b \in B} B(b, \epsilon_b/2).$$

Then U and B are open sets containing A and B and they are disjoint by the triangle inequality. □

Theorem 46. *Every well-ordered set X is normal in the order topology.*

4 Compactness

4.1 Compact Spaces

Definition 47 (Covering)**.** Given a space X, a *covering of* X is a collection $\mathcal{A}$ of sets of X whose union is X. It is called an *open covering* if $\mathcal{A}$ is a collection of open sets.

Definition 48 (Compact)**.** A space X is called *compact* if every open covering $\mathcal{A}$ of X contains a finite subcollection which is also an open cover of X.

Lemma 47 (Subspace Compactness Criterion)**.** *Let Y be a subspace of X. Then Y is compact if and only if every covering of Y by sets open in X contains a finite subcollection covering Y.*

Theorem 48. *Every closed subspace of a compact space is compact.*

Lemma 49 (Stronger Hausdorff Condition)**.** *If Y is a compact subspace of the Hausdorff space X and x_0 is not in Y, then there exist disjoint open sets U and V of X conatining x_0 and Y, respectively.*

Proof. For each point y of Y, choose disjoint neighborhoods U_y and V_y of the points x_0 and y, respectively (which we do using the Hausdorff condition). Since the collection $\{V_y \mid y \in Y\}$ is a covering of Y by sets open in X; therefore, finitely many of them $V_{y_1}, \ldots, V_{y_n}$ cover Y. Then the open set $V = V_{y_1} \cup \cdots \cup V_{y_n}$ contains Y and is disjoint from the open set $U = U_{y_1} \cap \cdots \cap U_{y_n}$, which contains x. □

Corollary 49.1. *Every compact subspace of a Hausdorff space is closed.*

Proof. To show that $X \setminus Y$ is open, it suffices to note that each point $x \in X \setminus Y$ has a neighborhood contained in $X \setminus Y$ by the above lemma; taking the union of these neighborhoods gives the desired result. □

Theorem 50 (Image of Compact Space)**.** *The image of a compact space under a continuous map is compact.*

Theorem 51 (Homeomorphism Condition)**.** *Let $f : X \to Y$ be a bijective continuous function. If X is compact and Y is Hausdorff, then f is homeomorphism.*

Theorem 52. *The product of finitely many compact spaces is compact.*

In fact, the following stronger fact is also true. However, we will prove it later.

Theorem 53 (Tychonoff's Theorem)**.** *The product of any collection of compact spaces is compact under the product topology.*

Definition 49 (Finite Intersection Property)**.** A collection $\mathscr{C}$ of subsets of X has the *finite intersection property* if for every finite subcollection $\{C_1, \dots, C_n\}$ of $\mathscr{C}$, the intersection $C_1 \cap \cdots \cap C_n$ is nonempty.

Theorem 54 (Alternative Compactness Criterion)**.** *Let X be a topological space. Then X is compact if and only if for every collection $\mathscr{C}$ of closed sets in X having the finite intersection property, the intersection $\bigcap_{C \in \mathscr{C}} C$ of all the elements of $\mathscr{C}$ is nonempty.*

Proof. Given a collection $\mathcal{A}$ of subsets of X, let

$$\mathscr{C} = \{X \setminus A \mid A \in \mathcal{A}\}$$

be the collection of their complements. Then the following statements hold:

1. $\mathcal{A}$ is a collection of open sets if and only if $\mathscr{C}$ is a collection of closed sets.
2. The collection $\mathcal{A}$ covers X if and only if the intersection $\bigcap_{C \in \mathscr{C}} C$ of all the elements of $\mathscr{C}$ is empty.
3. The finite subcollection $\{A_1, \dots, A_n\}$ of $\mathcal{A}$ covers X if and only if the intersection of the corresponding complements $C_i = X \setminus A_i$ of $\mathscr{C}$ is empty.

Now, notice that X is compact if and only if "given any collection $\mathcal{A}$ of open sets, if no finite subcollection of $\mathcal{A}$ covers X, then $\mathcal{A}$ does not cover X." Then, given the definition of $\mathscr{C}$ above, this statement is equivalent to stating that "given any collection $\mathscr{C}$ of closed sets, if every finite intersection of elements of $\mathscr{C}$ is nonempty, then the intersection of all the elements of $\mathscr{C}$ is nonempty." But this is just the condition of our theorem. □

Theorem 55. *Every compact Hausdorff space is normal.*

Proof. Regularity is given by Lemma 49. Then, given disjoint closed sets A and B in X choose, for each point a and A, disjoint open sets U_a and V_a containing a and B respectively (by regularity of X). The collection $\{U_a\}$ covers A; because A is compact, A may be covered by finitely many sets $U_{a_1}, \dots, U_{a_m}$. Then $U = U_{a_1} \cup \cdots \cup U_{a_m}$ and $V = V_{a_1} \cap \cdots \cap V_{a_m}$ are disjoint open sets containing A and B respectively. □

Definition 50 (Lindelöf Space)**.** A topological space X is called a *Lindelöf space* if every open cover has a countable subcover.

Definition 51 (σ-Compact)**.** A topological space X is called *σ-compact* if it is the union of countably many compact subspaces.

Theorem 56. *Any σ-compact space is Lindelöf.*

Proof. Let X be σ-compact and covered by some collection $\{U_\lambda\}_{\lambda \in \Lambda}$. Yet X is the union of countably many compact subspaces $C_1, C_2, C_3, \dots$. Each of these subspaces is covered by a finite subcover of $\{U_\lambda\}_{\lambda \in \Lambda}$, by the definition of compactness. Taking the union of countably many finite subcovers gives a countable subcover of all of X. □

4.2 Compact Subspaces of $\mathbb{R}$

Theorem 57 (Compactness of Closed Intervals)**.** *Let X be a totally ordered set having the least upper bound property. In the order topology, each closed interval in X is compact.*

Proof. Given $a < b$, let $\mathcal{A}$ be a covering of $[a,b]$ by sets open in $[a,b]$ in the subspace topology (which is the same as the order topology). We wish to prove the existence of a finite subcollection of $\mathcal{A}$ covering $[a,b]$. First we prove the following: if x is a point of $[a,b)$, then there is a point $y > x$ of $[a,b]$ such that the interval $[x,y]$ can be covered by at most two elements of $\mathcal{A}$.

If x has an immediate successor in X, then y is this immediate successor. Then $[x,y]$ consists of the two points x and y, so that it can be covered by at most two elements of $\mathcal{A}$. Otherwise, if x has no immediate successor in X, choose $A \in \mathcal{A}$ containing x. Because $x \neq b$ and A is open, A contains an interval of the form $[x,c)$ for some c in $[a,b]$. Choose a point y in (x,c); then the interval $[x,y]$ is covered by A.

Let C be the set of all points $y > a$ of $[a,b]$ such that the interval $[a,y]$ can be covered by finitely many elements of $\mathcal{A}$. Applying Step 1 to the case $x = a$, we see that there exists at least one such y, so C is nonempty. Let c be the least upper bound of the set C, so that $a < c \leq b$.

Now, we show that c belongs to C; that is, we show that the interval $[a,c]$ can be covered by finitely many elements of $\mathcal{A}$. Choose an element A of $\mathcal{A}$ containing c. Since A is open, it contains an interval of the form $(d,c]$ for some d in $[a,b]$. If c is not in C, there must be a point z of C lying in the the interval (d,c), because otherwise d would be smaller upper bound on C. Since z is in C, the interval $[a,z]$ can be covered by finitely many elements of $\mathcal{A}$. But $[z,c]$ lies in a single element A of $\mathcal{A}$, so $[a,c]$ can be covered by finitely many elements of $\mathcal{A}$, so $c \in C$, a contradiction with the assumption that $c \notin C$.

Finally, all we need to do is show that $c = b$. Suppose, therefore, for the sake of contradiction that $c < b$. Then by our earlier work, there exists a point $y > c$ of $[a,b]$ such that the interval $[c,y]$ can be covered by finitely many elements of $\mathcal{A}$. But $[a,c]$ can be covered by finitely many elements of $\mathcal{A}$, so the interval $[a,y]$ can also be covered by finitely many elements of $\mathcal{A}$. Yet this implies that y is in C, contradicting the fact that c is an upper bound on C. Therefore, our assumption $c < b$ must have been wrong and $c = b$. ☐

Corollary 57.1. *Every closed interval in $\mathbb{R}$ is compact.*

Theorem 58 (Characterizing Compact Subspaces of $\mathbb{R}^n$)**.** *A subspace A of $\mathbb{R}^n$ is compact if and only if it is closed and is bounded by the Euclidean metric d or the square metric ρ.*

Proof. It suffices to consider the square metric alone, since a set is bounded in the Euclidean metric if and only if it is bounded in the square metric.

Now, suppose that A is compact. Then it is closed by Corollary 49.1, since $\mathbb{R}^n$ is Hausdorff. Then consider the collection of open balls centered at the origin with radius m for each positive integer m. The union of these balls is all of $\mathbb{R}^n$, so some finite subcollection covers A. Hence some open ball centered at the origin with radius $M \in \mathbb{Z}^+$ covers A, so A is bounded under ρ with diameter at most $2M$.

Conversely, suppose that A is closed and bounded under ρ. Suppose that $\rho(x,y) \leq N$ for every pair x,y of points of A. Choose a point x_0 of A, and let $\rho(x_0,\mathbf{0}) = b$. The triangle inequality implies that $\rho(x,\overline{0}) \leq N+b$ for every x in A. Therefore, if $P = N+b$, then A is a subset of the cube $[-P,P]^n$, which is compact as the product of compact spaces. Therefore, as a closed subset of a compact space, A is compact. ☐

Theorem 59 (Extreme Value Theorem)**.** *Let $f : X \to Y$ be continuous, where Y is an totally ordered set with the order topology. If X is compact, then there exists points c and d in X such that $f(c) \leq f(x) \leq f(d)$ for every $x \in X$.*

Proof. Since f is continuous and X is compact, the set $A = f(X)$ is compact. We show that A has a largest element M and a smallest element m. Then since m and M belong to A, we must have $m = f(c)$ and $M = f(d)$ for some points c and d of X. Now, if A has no largest element, then the collection $\{(-\infty,a) \mid a \in A\}$

forms an open covering of A. Since A is compact, some finite subcollection $\{(-\infty, a_1), \ldots, (-\infty, a_n)\}$ covers A. If a_i is the largest of $a_1, \ldots, a_n$, then a_i belongs to none of these sets, contrary to the fact that they cover A. A dual argument shows that A has a smallest element. □

Definition 52 (Distance From a Set)**.** Let (X, d) be a metric space and $A \subseteq X$ be nonempty. For each $x \in X$, the *distance from x to A* is given by the equation $d(x, A) = \inf\{d(x, a) \mid a \in A\}$. For a fixed nonempty subset A, the function $x \mapsto d(x, A)$ is continuous.

Lemma 60 (Lebesgue Number Lemma)**.** *Let $\mathcal{A}$ be an open covering of the metric space (X, d). Then, if X is compact, there is a $\delta > 0$ such that for each subset of X having diameter less than δ, there exists an element of $\mathcal{A}$ containing it. The number δ is called a* Lebesgue number *for the covering $\mathcal{A}$.*

Proof. Let $\mathcal{A}$ be an open covering of X. If X itself is an element of $\mathcal{A}$, then any positive number is a Lebesgue number for $\mathcal{A}$. So assume X is not an element of $\mathcal{A}$. Choose a finite subcollection $\{A_1, \ldots, A_n\}$ of $\mathcal{A}$ that covers X. For each i, set $C_i = X \setminus A_i$ and define $f : X \to \mathbb{R}$ by letting $f(x)$ be the average of the numbers $d(x, C_i)$. That is, $f(x) = \frac{1}{n}\sum_{i=1}^{n} d(x, C_i)$.

Now, f is strictly positive – it can only be 0 if $x \in X$ is contained in C_i for every i, which would imply that x is not contained in A_i for every i, a contradiction with the assumption that A_i covers X. Furthermore, since it is continuous, it has a minimum value δ by Theorem 59.

Let $B \subseteq X$ have diameter less than δ. Choose a point x_0 of B, so that B lies in the δ-neighborhood of x_0. Now, $\delta \leq f(x_0) \leq d(x_0, C_m)$ where $d(x_0, C_m)$ is the largest of the numbers $d(x_0, C_i)$. But then the δ-neighborhood of x_0 is contained in the element $A_m = X \setminus C_m$ of the covering $\mathcal{A}$. □

Definition 53 (Uniformly Continuous)**.** A function $f : X \to Y$ of metric spaces is said to be *uniformly continuous* if given $\epsilon > 0$, there is a $\delta > 0$ such that for every pair of points $x_0, x_1 \in X$,

$$d_X(x_0, x_1) < \delta \Rightarrow d_Y(f(x_0), f(x_1)) < \epsilon$$

Theorem 61 (Continuous Map from Compact Space is Uniform)**.** *Let $f : X \to Y$ be a continuous map of the compact metric space (X, d_X) to the metric space (Y, d_Y). Then f is uniformly continuous.*

Proof. Given $\epsilon > 0$, take the open covering of Y by balls $B(y, \epsilon/2)$ of radius $\epsilon/2$. Let $\mathcal{A}$ be the open covering of X by the inverse images of these balls under f. Let δ be the Lebesgue number for $\mathcal{A}$. Then if x_1, x_2 are two points of X such that $d_X(x_1, x_2) < \delta$, the two-point set $\{x_1, x_2\}$ has diameter less than δ, so that its image $\{f(x_1), f(x_2)\}$ lies in some ball $B(y, \epsilon/2)$. Then $d_Y(f(x_1), f(x_2)) < \epsilon$. □

Definition 54 (Isolated Point)**.** If X is a space, a point of X is said to be an *isolated point* of X if the one-point $\{x\}$ is open in X.

Theorem 62 (Compact Hausdorff Space w/o Isolated Points are Uncountable)**.** *Let X be a nonempty compact Hausdorff space. If X has no isolated points, then X is uncountable.*

Proof. We will first start by proving the provisionary result that given any nonempty open set U of X and any point x of X, there exists a nonempty open set $V \subseteq U$ such that $x \notin \overline{V}$.

First, choose $y \in U \setminus x$. This must be possible since otherwise x would be an isolated point. Then, by the Hausdorff condition, we may choose disjoint open sets W_1 and W_2 containing x and y, respectively. Then $V = W_2 \cap U$ suffices; it is contained in U, and it is contained in the closed set $U \setminus (U \cap W_1)$, which does not contain x since $x \in W_1$. Hence the closure of V also does not contain x.

Now consider any labeling $x_1, x_2, \ldots$ of the points of X. We will show that there is a point of X missing from this sequence. Now, apply our provisionary result to the nonempty open set X to choose a nonempty set $V_1 \subseteq X$ such that $\overline{V}_1$ does not contain x_1. In general, given V_{n-1} open and nonempty, choose V_n to be a nonempty open set such that $V_{n-1} \supseteq V_n$ and $x_n \notin \overline{V_n}$. Therefore, we have a nested sequence

$$\overline{V_1} \supseteq \overline{V_2} \supseteq \cdots$$

of nonempty closed sets of X. Yet any finite collection of these nonempty closed sets has nonempty intersection, since they form a chain. Therefore, by Theorem 54, compactness implies that $M = \bigcap_n \overline{V_n}$ is nonempty. Yet M does not contain any of $x_1, x_2, \ldots$, whence the labeling is incomplete. □

Corollary 62.1. *Every closed interval in* $\mathbb{R}$ *is uncountable.*

4.3 Limit Points and Local Compactness

Definition 55 (Limit Point Compact)**.** A space X is said to be *limit point compact* if every infinite subset of X has a limit point.

Theorem 63. *Compactness implies limit point compactness but not conversely.*

Proof. Let X be a compact space. Suppose that A has no limit point. Then A contains all its limit points (trivially) so A is closed. Furthermore, for each $a \in A$ we can choose a neighborhood U_a of a such that U_a intersects A in the point a alone (otherwise, if we couldn't find such a neighborhood, $a \in A$ would be a limit point). The space X is covered by the open set $X \setminus A$ and the open sets U_a; being compact, it can be covered by finitely many of these sets. Since $X \setminus A$ does not intersect A, and each set U_a contains only one point of A, the set A must be finite.

For the counterexample, let $Y = \{y_1, y_2\}$ consist of two points and give Y the trivial topology. Then the space $X = Z_+ \times Y$ is limit point compact, for every (not just every *infinite*) nonempty subset of X has a limit point. In particular, (n, y) is the limit point of the set $\{(n, y)\}$ for any $(n, y) \in X$. However, it is not compact, for the covering of X by the open sets $U_n = \{n\} \times Y$ has no finite subcollection covering X. □

Definition 56 (Sequentially Compact)**.** Let X be a topological space. If (x_n) is a sequence of points of X, and if $n_1 < n_2 < \cdots < n_i < \cdots$ is an increasing sequence of positive integers, then the sequence (y_i) defined by setting $y_i = x_{n_i}$ is called a *subsequence* of the sequence (x_n). The space X is said to be *sequentially compact* if every sequence of points of X has a convergent subsequence.

Theorem 64 (Notions of Compactness)**.** *Let X be a metrizable space. Then the following are equivalent:*

1. *X is compact.*
2. *X is limit point compact.*
3. *X is sequentially compact.*

Definition 57 (Locally Compact at x)**.** A space X is said to be *locally compact at x* if there is some compact subspace C of X that contains a neighborhood of x. If X is locally compact at each of its points, X is said simply to be *locally compact*.

Theorem 65 (Compactification)**.** *Let X be a space. Then X is locally compact and Hausdorff if and only if there exists a space Y such that X is a subspace of Y, $Y \setminus X$ consists of a single point, and Y is a compact Hausdorff space. This space satisfies the following universal property: if Y and Y' are two spaces satisfying these conditions, then there is a homeomorphism of Y with Y' that equals the identity map on X.*

Proof. Uniqueness is obvious. Now, suppose that X is locally compact and Hausdorff. Let ∞ be some object which is not a point in X, and define $Y = X \cup \{\infty\}$. Topologizing Y by defining the collection of open sets of Y to consist of (1) all sets U that are open in X, and (2) all sets of the form $Y \setminus C$, where C is a compact subspace of X. It is not difficult to check that this is a topology on Y.

Now, we will show that X is a subspace of Y. Given any open set U of Y of type (1), its intersection with X is trivially an open set. On the other hand, if U is an open set of Y of type (2), then $(Y \setminus C) \cap X = X \setminus C$ is open. Conversely, any set open in X is a set of type (1) and therefore open in Y by definition.

Now, to show that Y is compact, let $\mathcal{A}$ be an open covering of Y. The collection $\mathcal{A}$ must contain an open set of type (2), say $Y \setminus C$, since no open set of type (1) contains ∞. Take all the members of $\mathcal{A}$ different from $Y \setminus C$ and intersect them with X; they form a collection of open sets of X covering C. Because C is compact, finitely many of them cover C; the corresponding finite collection of elements of $\mathcal{A}$ will, along with the element $Y \setminus C$, cover all of Y.

Now, we show that Y is Hausdorff. Let x and y be two distinct points of Y. If both of them lie in X, then we may apply the Hausdorff property of X to find disjoint neighborhoods. Therefore, assume that $y = \infty$. Then choose a compact set C in X containing a neighborhood U of x (by local compactness). Then U and $Y \setminus C$ are disjoint neighborhoods of x and $y = \infty$, respectively, in Y.

It remains to show the converse. Suppose Y is a compact Hausdorff space containing X as a subspace such that $Y \setminus X$ is a single point ∞. X is Hausdorff because it is a subspace of a Hausdorff space Y. Now take $x \in X$: we seek to show that X is locally compact at x. Choose disjoint open sets U and V of Y containing x and ∞, respectively. Then the set $C = Y \setminus V$ is closed in Y, so it is a compact subspace of Y by Theorem 48. Since C lies in X, it is also compact as a subspace of X, and contains the neighborhood U of x. □

Definition 58 (Compactification)**.** If Y is a compact Hausdorff space and X is a proper subspace of Y whose closure equals Y, then Y is a *compactification* of X. If $Y \setminus X$ equals a single point, then Y is called the *one-point compactification* of X.

Theorem 66 (Alternative Definition of Local Compactness)**.** *Let X be a Hausdorff space. Then X is locally compact if and only if given x in X, and given a neighborhood U of x, there is a neighborhood V of x such that $\overline{V}$ is compact and $\overline{V} \subseteq U$.*

Proof. Clearly this new formulation implies local compactness. For the converse, suppose X is locally compact. Then let $x \in X$ have neighborhood U. Take the one-point compactification Y of X and let C be the set $Y \setminus U$. Then C is closed in Y, so that C is a compact subspace of Y. Apply Lemma 49 to choose disjoint open sets V and W containing x and C, respectively. Then the closure $\overline{V}$ of V in Y is compact; furthermore, $\overline{V}$ is disjoint from C, so that $\overline{V} \subseteq U$, as desired. □

Corollary 66.1 (Subspace Locally Compact Criterion)**.** *Let X be locally compact and Hausdorff and let A be a subspace of X. Then if A is closed in X or open in X, A is locally compact.*

Proof. Suppose A is closed in X. Then given $x \in A$, let C be a compact subspace of X containing the neighborhood U of x in X. Then $C \cap A$ is closed in C and thus compact, and it contains the neighborhood $U \cap A$ of x in A. (We have not used the Hausdorff condition here.)

Suppose that A is open in X. Given $x \in A$, we apply the preceding theorem to choose a neighborhood V of x in X such that $\overline{V}$ is compact and $\overline{V} \subseteq A$. Then $C = \overline{V}$ is a compact subspace of A containing the neighborhood V of x in A. □

Corollary 66.2. *A space X is homeomorphic to an open subspace of a compact Hausdorff space if and only if X is locally compact and Hausdorff.*

Proof. Theorem 65 and Corollary 66.1. □

4.4 Tychonoff's Theorem

Lemma 67 (Tychonoff Lemma 1)**.** *Let X be a set; let $\mathcal{A}$ be a collection of subsets of X having the finite intersection property. Then there is a collection $\mathcal{D}$ of subsets of X such that $\mathcal{D}$ contains $\mathcal{A}$, has the finite intersection property, and is maximal with respect to the above two properties.*

Proof. A simple application of Zorn's Lemma. □

Lemma 68 (Tychonoff Lemma 2)**.** *Let X be a set and $\mathcal{D}$ a collection of subsets of X which is maximal with respect to the finite intersection property. Then:*

1. *Any finite intersection of elements of $\mathcal{D}$ is an element of $\mathcal{D}$.*

2. *If A is a subset of X that intersects every element of $\mathcal{D}$, then A is an element of $\mathcal{D}$.*

Proof. Both proofs proceed by assuming that the desired set is not contained in $\mathcal{D}$, and then showing that adding the desired set does not affect the finite intersection property, thereby drawing a contradiction with the maximality of $\mathcal{D}$. Details are left to the reader. □

Theorem 69. *An arbitrary product of compact spaces is compact in the product topology.*

Proof. Let $X = \prod_{\lambda \in \Lambda} X_\lambda$ where each space X_λ is compact. Let $\mathcal{A}$ be a collection of subsets of X having the finite intersection property. We prove that the intersection

$$\bigcap_{A \in \mathcal{A}} \overline{A}$$

is nonempty, and therefore by Theorem 54, X is compact.

First, use Lemma 67 to reduce the problem to a set $\mathcal{D}$ maximal with respect to the finite intersection property. Now, given $\lambda \in \Lambda$, let $\pi_\lambda : X \to X_\lambda$ be the projection map, and consider the collection

$$D_\lambda = \{\pi_\lambda(D) \mid D \in \mathcal{D}\}.$$

Now, this collection has the finite intersection property because $\mathcal{D}$ does. Yet by the compactness of X_λ, we can for each λ choose a point x_λ of X_λ such that $x_\lambda \in \bigcap_{D \in \mathcal{D}} \pi_\lambda(D)$.

Let $\mathbf{x}$ be the point $(x_\lambda)_{\lambda \in \Lambda}$ of X. Our goal is to show that $\mathbf{x} \in \overline{D}$ for every $D \in \mathcal{D}$; then our proof is finished. First, let $\pi_\beta^{-1} U_\beta$ be a subbasis element for the product topology on X containing $\mathbf{x}$. Now, the set U_β is a neighborhood of x_β in X_β, and since $x_\beta \in \overline{\pi_\beta(D)}$ by definition, U_β intersects $\pi_\beta(D)$ in some point $\pi_\beta(\mathbf{y})$, where $\mathbf{y} \in D$. Then it follows that $\mathbf{y} \in \pi_\beta^{-1}(U_\beta) \cap D$. In summary, any subbasis element containing $\mathbf{x}$ intersects every element of D.

But then, by Lemma 68, every subbasis element containing $\mathbf{x}$ belongs to $\mathcal{D}$. Again, by Lemma 68, every *basis* element containing $\mathbf{x}$ belongs to $\mathcal{D}$. Since $\mathcal{D}$ has the finite intersection property, this means that every basis element containing $\mathbf{x}$ intersects every element of $\mathcal{D}$; hence $\mathbf{x} \in \overline{D}$ for every $D \in \mathcal{D}$, as desired. □

4.5 The Stone-Čech Compatification

Definition 59 (Compactification)**.** A *compactification* of a space X is a compact Hausdorff space Y containing X as a subspace such that $\overline{X} = Y$. Two compactifications Y_1 and Y_2 of X are *equivalent* if there is a homeomorphism $h : Y_1 \to Y_2$ that pointwise fixes X.

If X has a compactification Y, then X must be completely regular, being a subspace of the completely regular space Y. Conversely, if X is completely regular, then X has a compactification. To see why, notice that X can be embedded in the compact Hausdorff space $[0, 1]^J$ for some J, and any such embedding gives rise to a compactification of X, as the following lemma shows:

Lemma 70. *Let X be a space and suppose that $h : X \to Z$ is an embedding of X in the compact Hausdorff space Z. Then there exists a corresponding compactification Y of X such that there is an embedding $H : Y \to Z$ that equals h to X. The compactification Y is uniquely determined up to equivalence.*

Proof. Given h, let X_0 denote the subspace $h(X)$ of Z and let Y_0 denote its cloure in Z. Then Y_0 is a compact Hausdorff space and $\overline{X_0} = Y_0$; therefore, Y_0 is a compactification of X_0.

Now, we will construct a space Y containing X such that the pair (X, Y) is homeomorphic to the pair (X_0, Y_0). Let us choose a set A disjoint from X that is in bijective correspondence with the set $Y_0 \setminus X_0$ under some map $k : A \to Y_0 \setminus X_0$. Define $Y = X \cup A$, so there is a natural bijection $H : Y \to Y_0$.

Next, we topologize Y by declaring U to be open in Y if and only if $H(U)$ is open in Y_0. The map H is automatically a homeomorphism, and the space X is a subspace of Y because H equals to the homeomorphism h when restricted to the subspace X of Y. Also, by expanding the range of H, we obtain the required embedding of Y into Z.

Proving that Y is uniquely determined up to equivalence is simple and left to the reader. □

Theorem 71 (The Existence of Compactifications Extending Bounded Continuous Maps)**.** *Let X be a completely regular space. There exists a compactification Y of X having the property that every bounded continuous map $f : X \to \mathbb{R}$ extends uniquely to a continuous map of Y into $\mathbb{R}$.*

Proof. Let $\{f_\alpha\}_{\alpha \in J}$ be the collection of *all* bounded continuous real-valued functions on X. For each $\alpha \in J$, define $I_\alpha = [\inf f_\alpha(X), \sup f_\alpha(X)]$. Then define $h : X \to \prod_{\alpha \in J} I_\alpha$ by $h(x) = (f, \alpha(x))_{\alpha \in J}$. By the Tychonoff theorem, $\prod I_\alpha$ is compact. Because X is completely regular, the collection $\{f_\alpha\}$ seperates points from closed sets in X. Therefore, h is an embedding.

Let Y be the compactification of X induced by the embedding h. Then there is an embedding $H : Y \to \prod I_\alpha$ that equals h when restricted to the subspace X of Y. Given a bounded continuous real-valued function f on X, we show it extends to Y. The function f belongs to the collection $\{f_\alpha\}_{\alpha \in J}$, so it equals f_β for some index β. Let $\pi_\beta : \prod I_\alpha \to I_\beta$ be the projection mapping. Then the continuous map $\pi_\beta \circ H : Y \to I_\beta$ is the desired extension of f. For if $x \in X$,

$$\pi_\beta(H(X)) = \pi_\beta(h(x)) = \pi_\beta((f_\alpha(x))_\alpha \in J) = f_\beta(x).$$

□

Uniqueness of the extension follows from the following lemma:

Lemma 72. *Let $A \subseteq X$ and $f : A \to Z$ be a continuous map of A into a Hausdorff space Z. Then there is at most one extension of f to a continuous function $g : \overline{A} \to Z$.*

Proof. Suppose $g, g' : \overline{A} \to X$ are two different extensions of f; choose x so that $g(x) \neq g'(x)$. Then, let U and U' be disjoint neighborhoods of $g(x)$ and $g(x')$ respectively. Choose a neighborhood V of x so that $g(V) \subseteq U$ and $g'(V) \subseteq U'$. Now V intersects A in some point y; then $g(y) \in U$ and $g'(y) \in U'$. But since $y \in A$, we have $g(y) = f(y)$ and $g'(y) = f(y)$, which contradictions the fact that U and U' are disjoint. □

Theorem 73. *Let X be a completely regular space such that Y is a compactification of X satisfying the extension property of Theorem 71. Then, given any continuous map $f : X \to C$ of X into a compact Hausdorff space C, the map f extends uniquely a continuous map $g : Y \to C$.*

Proof. Since C is completely regular, it can be embedded in $[0, 1]^J$ for some J. Hence assume $C \subseteq [0, 1]^J$. Then each component function f_α of the map f is a bounded continuous real-valued function on X; by hypothesis, f_α can be extended to a continuous map g_α of Y into $\mathbb{R}$. Define $g : Y \to \mathbb{R}^J$ by setting $g(y) = (g_\alpha(y))_{\alpha \in J}$; then g is continuous because $\mathbb{R}^J$ has the product topology. Now g maps Y into C, as continuity of g implies $g(Y) = g(\overline{X}) \subseteq \overline{g(X)} = \overline{f(X)} \subseteq \overline{C} = C$. Hence g is the desired extension of f. □

Theorem 74. *Let X be a completely regular space. If Y_1 and Y_2 are two compacifications of X satisfying the extension property of Theorem 38.2, then Y_1 and Y_2 are equivalent.*

Proof. The usual proof for universal properties. □

Definition 60 (Stone-Čech Compacification)**.** For every completely regular space X, there is a unique compactification $\beta(X)$ of X, called the *Stone-Čech Compacification of X* satisfying the extension condition of Theorem 71. It is uniquely characterized by the fact that any continuous map $f : X \to C$ of X into a compact Hausdorff space C extends uniquely to a continuous map $g : \beta(X) \to C$.

4.6 Embedding Compact Manifolds in $\mathbb{R}^N$

Definition 61 (Manifold)**.** An m-manifold is a Hausdorff space X with a countable basis such that each point x of X has a neighborhood that is homeomorphic with an open subset of $\mathbb{R}^m$. A 1-manifold is called a *curve* and a 2-manifold is called a *surface*.

Definition 62 (Support)**.** If $\phi : X \to \mathbb{R}$ is a function, then the *support* of ϕ (denoted $\operatorname{Supp}\phi$) is defined to be the closure of the set $\phi^{-1}(R \setminus \{0\})$. Thus, if x lies outisde the support of ϕ, there is some neighborhood of x on which ϕ vanishes.

Definition 63 (Partition of Unity)**.** Let $\{U_1, \dots, U_n\}$ be a finite indexed open covering of the space X. An indexed family of continuous functions $f_i : X \to [0, 1]$ for $i = 1, \dots, n$ is said to be a *partition of unity dominated by* $\{U_i\}$ if $\operatorname{Supp}\phi_i \subseteq U_i$ for each i and $\sum_{i=1}^n \phi_i(x) = 1$ for each x.

Theorem 75 (Existence of Finite Partitions of Unity)**.** *Let $\{U_1, \dots, U_n\}$ be a finite open covering of the normal space X. Then there exists a partition of unity dominated by $\{U_i\}$.*

Proof. We begin with the following lemma: one can "shrink" the covering $\{U_i\}$ to an open covering $\{V_1, \dots, V_n\}$ of X such that $\overline{V}_i \subseteq U_i$ for each i.

First, note that the set $X \setminus (U_2 \cup \cdots \cup U_n)$ is a closed subset of X. Because $\{U_1, \dots, U_n\}$ covers X, A is contained in U_1. Then, by normality, there exists an open set containing A such that $\overline{V}_1 \subseteq U_1$. Then the collection $\{V_1, U_2, \dots, U_n\}$ covers X. In general, given open sets $V_1, \dots, V_{k-1}$ such that the collection $\{V_1, \dots, V_{k-1}, U_k, \dots, U_n\}$ covers X, we let $A = X \setminus (V_1 \cup \cdots \cup V_{k-1} \cup U_{k+1} \cup \cdots \cup U_n)$. Then A is a closed subset of X containing in the open set U_k. Choose V_k to be an open set containing A such that $\overline{V}_k \subseteq U_k$. Then $\{V_1, \dots, V_k, U_{k+1}, \dots, U_n\}$ covers X. At the nth step of the induction, the lemma is shown.

With this lemma, we are ready to prove the theorem. Given the open covering $\{U_1, \dots, U_n\}$, we use the lemma to choose an open covering $\{V_1, \dots, V_n\}$ of X such that $\overline{V}_i \subseteq U_i$ for each i. Then choose an open covering $\{W_1, \dots, W_n\}$ of X such that $\overline{W}_i \subseteq V_i$ for each i. Using the Urysohn lemma, choose for each i a continuous function $\psi_i : X \to [0, 1]$ such that $\psi_i(\overline{W}_i) = \{1\}$ and $\psi_i(X \setminus V_i) = \{0\}$. Since $\psi_i^{-1}(R \setminus \{0\})$ is contained in V_i, we have $\operatorname{Supp}\psi_i \subseteq \overline{V}_i \subseteq U_i$. Because the collection $\{W_i\}$ covers X, the sum $\Psi(x) = \sum_{i=1}^n \psi_i(x)$ is positive for each x. Therefore, we may define, for each j,

$$\psi_j(x) = \frac{\psi_j(x)}{\Psi(x)}$$

which creates the desired partition of unity. $\square$

Theorem 76. *If X is a compact m-manifold, then X can be embedded in $\mathbb{R}^N$ for some positive integer N.*

Proof. Cover X by finitely many open sets $\{U_1, \dots, U_n\}$ each of which may be embedded in $\mathbb{R}^m$. Choose embeddings $g_i : U_i \to \mathbb{R}^m$ for each i. Being compact and Hausdorff, X is normal. Let $\phi_1, \dots, \phi_n$ be a partition of unity dominated by $\{U_i\}$; let $A_i = \operatorname{Supp}\phi_i$. Then, for each $i = 1, \dots, n$ define a function $h_i : X \to \mathbb{R}^m$ by the rule

$$h_i(x) = \begin{cases} \phi_i(x) \cdot g_i(x) & \text{for } x \in U_i \\ \mathbf{0} = (0, \dots, 0) & x \in X \setminus A_i. \end{cases}$$

The function h_i is well defined because the two definitions of h_i agree on the intersection of their domains, and h_i is continuous because its restrictions to the open sets U_i and $X \setminus A_i$ are continuous. Now define

$$F : X \to (\underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ times}} \times \underbrace{\mathbb{R}^m \times \cdots \times \mathbb{R}^m}_{n \text{ times}})$$

by the rule $F(x) = (\phi_1(x), \dots, \phi_n(x), h_1(x), \dots, h_n(x))$. Clearly F is continuous. Therefore, because X is compact, it suffices to show that F is injective. Suppose that $F(x) = F(y)$. Then $\phi_i(x) = \phi_i(y)$ and $h_i(x) = h_i(y)$ for all i. Now $\phi_i(x) > 0$ for some i [since $\sum \phi_x = 1$]. Therefore, $\phi_i(y) > 0$ also, so that $x, y \in U_i$. Then

$$\phi_i(x) \cdot g_i(x) = h_i(x) = h_i(y) = \phi_i(y) \cdot g_i(y).$$

$\phi_i(x) = \phi_i(y) > 0$ implies $g_i(x) = g_i(y)$. But $g_i : U_i \to \mathbb{R}^m$ is injective, so $x = y$, as desired. $\square$

5 More on Metric Spaces

This section, which contains many technical lemmas and theorems, omits many of the longer proofs and instead cites their proof in Munkres' Topology 2nd ed.

5.1 The Urysohn Lemma

Theorem 77 (Urysohn Lemma)**.** *Let X be a normal space; let A and B be disjoint closed subsets of X. Let $[a, b]$ be a closed interval in the real line. Then there exists a continuous map $f : X \to [a, b]$ such that $f(x) = a$ for every $x \in A$ and $f(x) = b$ for every $x \in B$.*

Proof. Pg. 207-210 of Munkres. □

Definition 64 (Seperated by Continuous Function)**.** If A and B are two subsets of the toplogical space X and there is a continuous function $f : X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$, we say that A and B *can be separated by a continuous function.*

The Urysohn Lemma states that if every pair of disjoint closed sets in X can be separated by disjoint open sets, then each such pair can be separated by a continuous function. The converse is also true, and in fact trivial: if $f : X \to [0, 1]$ is a continuous function separating A and B, then $f^{-1}([0, \frac{1}{2}))$ and $f^{-1}((\frac{1}{2}, 1])$ are disjoint open sets containing A and B, respectively.

Definition 65 (Completely Regular)**.** A space X is *completely regular* if one-point sets are closed in X and if for each point x_0 and each closed set A not containing x_0, there is a continuous function $f : X \to [0, 1]$ such that $f(x_0) = 1$ and $f(A) = \{0\}$.

Theorem 78 (Completely Regular Spaces)**.** *A subspace of a completely regular space is completely regular. A product of completely regular spaces is completely regular.*

5.2 Urysohn Metrization Theorem

Theorem 79 (Urysohn Metrization Theorem)**.** *Every regular space X with a countable basis is metrizable.*

Proof. Pg. 215-217 of Munkres. □

Theorem 80 (Tietze Extension Theorem)**.** *Let X be a normal space with closed subspace A. Then,*

1. *Any continuous map $A \to [a, b]$ may be extended to a continuous map $X \to [a, b]$ (where $[a, b] \subseteq \mathbb{R}$).*

2. *Any continuous map $A \to \mathbb{R}$ may be extended to a continuous map $X \to \mathbb{R}$.*

Proof. Pg. 219-222 of Munkres. □

5.3 The Nagata-Smirnov Metrization Theorem

Definition 66 (Locally Finite)**.** Let X be a topological space. Then $\mathcal{A}$, a collection of subsets of X, is *locally finite* in X if every point of X has a neighborhood that intersects only finitely many elements of $\mathcal{A}$.

Lemma 81 (Properties of Locally Finite)**.** *Let $\mathcal{A}$ be a locally finite collection of subsets of X. Then*

1. *Any subcollection of $\mathcal{A}$ is locally finite.*

2. *The collection $\mathcal{B} = \{\overline{A}\}_{A \in \mathcal{A}}$ of the closures of the elements of $\mathcal{A}$ is locally finite.*

3. $\overline{\bigcup_{A \in \mathcal{A}} A} = \bigcup_{A \in \mathcal{A}} \overline{A}$.

Proof. (1) is trivial, so consider (2). Notice that any open set U that intersects $\overline{A}$ intersects A. Therefore, if U is a neighborhood of x that intersects only finitely many elements A of $\mathcal{A}$, U can intersect at most the same number of sets of the collection $\mathcal{B}$.

For (3), let Y denote the union of the elements of $\mathcal{A}$. Now, in general $\bigcup \overline{A} \subseteq \overline{Y}$, so it suffices to prove the reverse inclusion. Let $x \in \overline{Y}$. Now, let U be a neighborhood of x that intersects only finitely many elements $A_1, \dots, A_k$ of $\mathcal{A}$. Now, x must belong to one of $\overline{A}_1, \dots, \overline{A}_k$ (and hence it must belong to the union $\bigcup \overline{A}$), since otherwise $U \setminus (\overline{A}_1 \cup \cdots \cup \overline{A}_k)$ would be a neighborhood of x that intersects no element of $\mathcal{A}$ and hence does not intersect Y, contrary to the assumption that $x \in \overline{Y}$. □

Definition 67 (Countably Locally Finite)**.** A collection $\mathcal{B}$ of subsets of X is *countably locally finite* or *σ-locally finite* if $\mathcal{B}$ can be written as the countable union of collections $\mathcal{B}_n$, each of which is locally finite.

Definition 68 (Refinement)**.** Let $\mathcal{A}$ be a collection of subsets of X. Then a collection $\mathcal{B}$ of subsets of X is said to be a *refinement* of $\mathcal{A}$ (or is said to *refine* $\mathcal{A}$) if for each element B of $\mathcal{B}$, there is an element A of $\mathcal{A}$ containing B. If the elements of $\mathcal{B}$ are open sets, we call $\mathcal{B}$ an *open refinement* of $\mathcal{A}$; if they are closed sets, we call $\mathcal{B}$ a *closed refinement.*

Lemma 82 (Countably Locally Finite Refinements)**.** *Let X be a metrizable space. If $\mathcal{A}$ is an open covering of X, then there is an open covering $\mathcal{E}$ of X refine $\mathcal{A}$ that is countably locally finite.*

Proof. Pg. 246-247 of Munkres. □

Definition 69 (G_δ Set)**.** A subset A of a space X is called a G_δ set in X if it equals the intersection of a countable collection of open subsets of X.

Lemma 83 (Regular Spaces with Countably Locally Finite Basis)**.** *Let X be a regular space with a basis $\mathcal{B}$ that is countably locally finite. Then X is normal, and every closed set in X is a G_δ set in X.*

Proof. Pg. 249-250 of Munkres. □

Lemma 84 (Normal Spaces with Closed G_δ Sets)**.** *Let X be normal; let A be a closed G_δ set in X. Then there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ for $x \in A$ and $f(x) > 0$ for $x \notin A$.*

Proof. Pg. 250 of Munkres. □

Theorem 85 (Nagata-Smirnov Metrization Theorem)**.** *A space X is metrizable if and only if X is regular and has a basis that is countably locally finite.*

Proof. Pg. 250-252 of Munkres. □

5.4 Paracompactness

Definition 70 (Paracompact)**.** A space X is *paracompact* if every open covering $\mathcal{A}$ of X has a locally finite open refinement $\mathcal{B}$ that covers X.

Theorem 86. *Every paracompact Hausdorff space is normal.*

Proof. First, we will prove regularity. Choose $a \in X$ and B a closed set of X not containing a. For each $b \in B$, choose an open set U_b about b whose closure is disjoint from a. Then, cover X by the open sets U_b alongside the open set $X \setminus B$. Let $\mathscr{C}$ be a locally finite open refinement of this covering which also covers X. Now let $\mathscr{D}$ be the subcollection of $\mathscr{C}$ consisting of every element of $\mathscr{C}$ that intersects B. Then $\mathscr{D}$ clearly covers B, but also if $D \in \mathscr{D}$ then $\overline{D}$ is disjoint from a. For D intersects B, so it lies in some set U_b whose closure is disjoint from a. Then $V = \bigcup_{D \in \mathscr{D}} D$ is an open set in X containing B. Since $\mathscr{D}$ is locally finite, by Lemma 81 we have that

$$\overline{V} = \bigcup_{D \in \mathcal{D}} \overline{D}$$

so that $\overline{V}$ is disjoint from a, and regularity is proven. From here, proving normality is simple: we repeat the same argument, replacing a by a closed set A and the Hausdorff condition by regularity. □

Theorem 87. *Every closed subspace of a paracompact space is paracompact.*

Proof. Let Y be a closed subspace of the paracompact space X and $\mathcal{A}$ be a covering of Y by sets open in Y. Then for each $A \in \mathcal{A}$, choose an open set A' of X such that $A' \cap Y = A$. Cover X by the open sets A', along with the open set $X \setminus Y$. Let $\mathcal{B}$ be a locally finite open refinement of this covering that covers X. The collection $\mathscr{C} = \{B \cap Y \mid B \in \mathcal{B}\}$ is the required locally finite open refinement of $\mathcal{A}$. □

Lemma 88. *[Equivalent Paracompact Conditions] Let X be regular. Then the following conditions on X are equivalent:*

Every open covering of X has a refinement that is:

1. *An open covering of X and countably locally finite.*
2. *A covering of X and locally finite.*
3. *A closed covering of X and locally finite.*
4. *An open covering of X and locally finite.*

Proof. Pg. 254-257 of Munkres. □

Theorem 89. *Every metrizable space is paracompact.*

Proof. Let X be a metrizable space. By Lemma 82, given an open covering $\mathcal{A}$ of X, it has an open refinement that covers X is *countably* locally finite. Then since any metrizable space is normal, and therefore regular, Lemma 88 implies that $\mathcal{A}$ has an open refinement that covers X and is locally finite. □

Theorem 90. *Every regular Lindelöf space is paracompact.*

Proof. Let X be regular and Lindelöf. Given an open covering $\mathcal{A}$ of X it has a countable subcollection that covers X; this subcollection is automatically countably locally finite. Then, we apply Lemma 88 to show $\mathcal{A}$ has an open refinement that covers X and is locally finite. □

5.5 Complete Metric Spaces

Definition 71 (Cauchy Sequences)**.** Let (X, d) be a metric space. A sequence (x_n) of points of X is said to be a *Cauchy sequence in* (X, d) if it has the property that given $\epsilon > 0$, there is an integer N such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$.

Definition 72 (Complete Metric Space)**.** A metric space (X, d) is said to be *complete* if every Cauchy sequence in X converges.

Lemma 91. *Let X be a complete metric space, and A be a closed subset of X. Then A under the induced metric is complete.*

Lemma 92. *A metric space X is complete if every Cauchy sequence in X has a convergent subsequence.*

Theorem 93. *For any $k \in \mathbb{Z}^+$, $\mathbb{R}^k$ is complete under the euclidean metric d and the square metric ρ.*

Definition 73 (Uniform Metric)**.** Let (Y, d) be a metric space and let $\overline{d}(a, b) = \min\{d(a, b), 1\}$ be the standard bounded metric on Y induced by d. Then, if $x = (x_\lambda)_{\lambda \in \Lambda}$ and $y = (y_\lambda)_{\lambda \in \Lambda}$ are points of the Cartesian product Y^Λ, then define the *uniform metric corresponding to* d on Y^Λ as so:

$$\overline{\rho}(x, y) = \sup\{\overline{d}(x_\lambda, y_\lambda) \mid \lambda \in \Lambda\}$$

Theorem 94 (Completeness of Uniform Metric Space)**.** *Suppose (Y, d) is a complete metric space. Then, for any set Λ, the space Y^Λ equipped with the uniform metric corresponding to d is also complete.*

Now, the Cartesian product Y^Λ can also be considered as the space of all functions $\Lambda \to Y$. A natural question is to consider the space of particular types of functions – for example continuous or bounded. However, for this, Λ cannot simply be a set, but also needs to be a topological space.

Definition 74 (Spaces of Continuous and Bounded Functions)**.** Let X be a topological space and (Y, d) be a metric space. Then $\mathscr{C}(X, Y)$ is the set of continuous functions $f : X \to Y$, and $\mathcal{B}(X, Y)$ is the set of bounded functions $f : X \to Y$ (that is, functions for which $f(X)$ is bounded).

Theorem 95 (Completeness of Function Spaces)**.** *Let X be a topological space and (Y, d) be a metric space. The set $\mathscr{C}(X, Y)$ of continuous functions is closed in Y^X under the uniform metric. So is the set $\mathcal{B}(X, Y)$ of bounded functions. Therefore, if Y is complete, these spaces are complete in the uniform metric.*

Definition 75 (sup Metric)**.** If (Y, d) is a metric space, one can define another metric, called the *sup metric* on the set $\mathcal{B}(X, Y)$ of bounded functions from X to Y by the equation

$$\rho(f, g) = \sup\{d(f(x), g(x)) \mid x \in X\}.$$

Notice that if $f, g \in \mathcal{B}(X, Y)$, then $\overline{\rho}(f, g) = \min\{\rho(f, g), 1\}$. That is, the uniform metric is just the standard bounded metric derived from the sup metric.

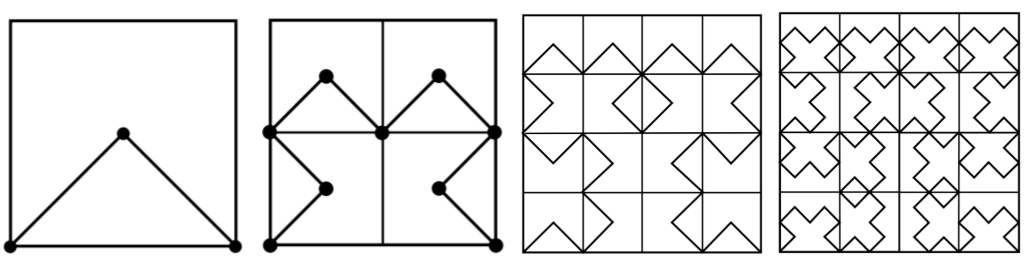
Theorem 96. *Let (X, d) be a metric space. There is an isometric embedding of X into a complete metric space, namely the set $\mathcal{B}(X, \mathbb{R})$ of all bounded functions $X \to \mathbb{R}$.*

Definition 76 (Completion)**.** Let X be a metric space. If $h : X \to Y$ is an isometric embedding of X into a complete metric space Y, then the subspace $\overline{h(X)}$ of Y is a complete metric space, called the *completion of* X, and it is uniquely determined up to isometry.

5.6 Space-Filling Curves

Theorem 97. *Let $I = [0, 1]$. Then there exists a continuous map $f : I \to I^2$ such that $\operatorname{im} f = I^2$.*

Proof. First, we construct a sequence $\{f_n\}$ of continuous "partial space-filling curves" $I \to I^2$ as so:



Let $d(x, y)$ denote the square metric on $\mathbb{R}^2$, and let ρ denote the corresponding sup metric on $\mathscr{C}(I, I^2)$. Because I^2 is closed in $\mathbb{R}^2$, it's complete by Lemma 91, whence $\mathscr{C}(I, I^2)$ is complete in ρ by Theorem 95.

Notice that because each of the small triangular paths that make up f_n lies in a square of edge length $1/2^n$ such that the operation replacing f_n with f_{n+1} only operates within said square (replacing each triangular path with 4 triangular paths), the distance between $f_n(t)$ and $f_{n+1}(t)$ is at most $1/2^n$. Hence $\{f_n\}$ is Cauchy, so it has a limit function $f : I \to I^2$. This function is continuous because it belongs to $\mathscr{C}(I, I^2)$.

Let $x \in I^2$ be an arbitrary point. Notice that, given n, the f_n comes within a distance of $1/2^n$ of the point x since it visits each of the small squares of edge length $1/2^n$. Using this fact, we prove that any ϵ-neighborhood of x intersects $f(I)$. Choose N large enough that

$$\rho(f_N, f) < \epsilon/2 \text{ and } 1/2^N < \epsilon^2.$$

Now, as previously discussed, there is a point $t_0 \in I$ such that $d(x, f_n(t_0) \leq 1/2^N$. Then, by the criterion above, $d(x, f(t_0)) < \epsilon$, so the ϵ-neighborhood of x intersects $f(I)$. Since any neighborhood of x intersects $f(I)$, it follows that x belongs to the closure of $f(I)$. But I is compact, so $f(I)$ is compact and therefore closed. Hence x lies in $f(I)$, as desired. ☐

5.7 Compactness in Metric Spaces

Now, compactness, limit point compactness, and sequential compactness are equivalent for metric spaces. Hence any compact metric space is complete.

Definition 77 (Totally Bounded)**.** A metric space (X, d) is *totally bounded* if for every $\epsilon > 0$, there is a finite covering of X by ϵ-balls.

Theorem 98 (Condition for Compactness)**.** *A metric space (X, d) is compact if and only if it is complete and totally bounded.*

Definition 78 (Equicontinuity)**.** Let (Y, d) be a metric space. Let $\mathcal{F}$ be a subset of the function space $\mathscr{C}(X, Y)$. If $x_0 \in X$ the set $\mathcal{F}$ of functions is siad to be *equicontinuous at* x_0 if given $\epsilon > 0$, there is a neighborhood U of x_0 such that for all $x \in U$ and all $f \in \mathcal{F}$, $d(f(x), f(x_0)) < \epsilon$. If $\mathcal{F}$ is equicontinuous at x_0 for each $x_0 \in X$, it is just *equicontinuous.*

Lemma 99. *Let X be a space and (Y, d) be a metric space. If the subset $\mathcal{F}$ of $\mathscr{C}(X, Y)$ is totally bounded under the uniform metric corresponding to d, then $\mathcal{F}$ is equicontinuous under d.*

Proof. Pg. 277 of Munkres. □

Lemma 100. *Let X be a compact space and (Y, d) be a compact metric space. Then if the subset $\mathcal{F}$ of $\mathscr{C}(X, Y)$ is equicontinuous under d, then $\mathcal{F}$ is totally bounded under the uniform and sup metrics corresponding to d.*

Proof. Pg. 277-278 of Munkres. □

Definition 79 (Pointwise Bounded)**.** If (Y, d) is a metric space, a subset $\mathcal{F}$ of $\mathscr{C}(X, Y)$ is *pointwise bounded* under d if for each $x \in X$, the subset $\mathcal{F}_x = \{f(x) \mid f \in \mathcal{F}\}$ of Y is bounded under d.

Theorem 101 (Ascoli's Theorem)**.** *Let X be a compact space; let $(\mathbb{R}^n, d)$ denote Euclidean space under the square metric or the Euclidean metric; give $\mathscr{C}(X, \mathbb{R}^n)$ the corresponding uniform topology. A subspace $\mathcal{F}$ of $\mathscr{C}(X, \mathbb{R}^n)$ has compact closure if and only if $\mathcal{F}$ is equicontinuous and pointwise bounded under d.*

Proof. Pg. 278-279 of Munkres. □

5.8 Pointwise and Compact Convergence

Definition 80 (Topology of Pointwise Convergence)**.** Given $x \in X$ and an open set U of the space Y, let

$$S(x, U) = \{f \mid f \in Y^X \text{ and } f(x) \in U\}.$$

The sets $S(x, U)$ are a subbasis for a topology on Y^X, called the *topology of pointwise convergence.*

Theorem 102 (Pointwise Convergence)**.** *A sequence f_n of functions converges to the function f in the topology of pointwise convergence if and only if for each $x \in X$, the sequence $f_n(x)$ of points of Y converges to the point $f(x)$.*

Definition 81 (Topology of Compact Convergence)**.** Let (Y, d) be a metric space and X be a topological space. Given an element $f \in Y^X$, a compact subspace C of X, and a number $\epsilon > 0$, let $B_C(f, \epsilon)$ denote the set of all those elements g of Y^X for which

$$\sup\{d(f(x), g(x)) \mid x \in C\} < \epsilon.$$

The sets $B_C(f, \epsilon)$ form a basis for a topology on Y^X, called the *topology of compact convergence.*

Theorem 103. *A sequence $f_n : X \to Y$ of functions converges to the function f in the topology of compact convergence if and only if, for each compact subspace C of X, the sequence $f_n|_C$ converges uniformly to $f|_C$.*

Definition 82 (Compactly Generated)**.** A space X is said to be *compactly generated* if it satisfies the following condition: a set A is open in X if $A \cap C$ is open in C for each compact subspace C of X.

Lemma 104 (Useful Criterion for Compact Generation)**.** *If X is locally compact or X satisfies the first countability axiom, then X is compactly generated.*

Proof. Suppose X is locally compact. Then let $A \cap C$ be open in C for every compact subspace C of X. Given $x \in A$, choose a neighborhood U of x that lies in a compcat subspace C of X. Since $A \cap C$ is open in C by hypothesis, $A \cap U$ is open in U¡ and hence open in X. Then $A \cap U$ is a neighborhood of x contained in A, so A is open in X.

Suppose that X satisfies the first countability axiom. It suffices to show that if $B \cap C$ is closed in C for each compact subspace C of X, then B is closed. Take $x \in \overline{B}$. Since X has a countable basis at x, there is a sequence (x_n) of points of B converging to x. The subspace $C = \{x\} \cup \{x_n \mid n \in \mathbb{Z}_+\}$ is compact, so that $B \cap C$ is by assumption closed in C. Since $B \cap C$ contains x_n for every n, it contains x as well. Therefore $x \in B$. This implies that $\overline{B} \subseteq B$, so B is closed. □

This definition is relevant for the following reason:

Lemma 105. *If X is compactly generated, then a function $f : X \to Y$ is continuous if for each compact subspace C of X, the restricted function $f|_C$ is continuous.*

Proof. Let $V \subseteq Y$ be open. Given any subspace C of X, $f^{-1}(V) \cap C) = (f|_C)^{-1}(V)$. If C is compact, this set is open in C because $f|_C$ is continuous. Since X is compactly generated, $f^{-1}(V)$ is open in X. □

Finally, the name is explained by the following theorems:

Theorem 106. *Let X be a compactly generated space and (Y, d) be a metric space. Then $\mathscr{C}(X, Y)$ is closed in Y^X in the topology of compact convergence.*

Corollary 106.1. *Let X be a compactly generated space; let (Y, d) be a metric space. If a sequence of continuous functions $f_n : X \to Y$ converges to f in the topology of compact convergence, then f is continuous.*

Theorem 107. *Let X be a space let (Y, d) be a metric space. For the function space Y^X, one has the following inclusions of topologies:*

$$\text{(uniform)} \supseteq \text{(compact convergence)} \supseteq \text{(pointwise convergence)}$$

If X is compact, the first two coincide, and if X is discrete, the second two coincide.

Definition 83. Let X and Y be topological spaces. If C is a compact subspace of X and U is an open subset of Y, define $S(C, U) = \{f \mid f \in \mathscr{C}(X, Y) \text{ and } f(C) \subseteq U\}$. The sets $S(C, Y)$ form a subbasis for a topology on $\mathscr{C}(X, Y)$ that is called the *compact-open topology.*

Theorem 108. *Let X be a space and let (Y, d) be a metric space. On the set $\mathscr{C}(X, Y)$, the compact-open topology and the topology of compact convergence coincide.*

Proof. Pg. 285-286 of Munkres. □

Corollary 108.1. *Let Y be a metric space. The compact convergence topology on $\mathscr{C}(X, Y)$ does not depend on the metric of Y. Therefore, if X is compact, the uniform topology on $\mathscr{C}(X, Y)$ does not depend on the metric of Y.*

Theorem 109. *Let X be locally compact Hausdorff; let $\mathscr{C}(X, Y)$ have the compact-open topology. Then the evaluation map $e : X \times \mathscr{C}(X, Y) \to Y$ given by $e(x, f) = f(x)$ is continuous.*

Proof. Pg. 287 of Munkres. □

Definition 84. Given a function $f : X \times Z \to Y$, there is a corresponding *induced function* $F : Z \to \mathscr{C}(X, Y)$ defined by the equation $(F(z))(x) = f(x, z)$. Conversely, given $F : Z \to \mathscr{C}(X, Y)$, this equation defines a corresponding function $f : X \times Z \to Y$. We say that F is the map of Z into $\mathscr{C}(X, Y)$ that is *induced* by f.

Theorem 110. *Let X and Y be spaces; give $\mathscr{C}(X, Y)$ the compact-open topology. If $f : X \times Z \to Y$ is continuous, then so is the induced function $F : Z \to \mathscr{C}(X, Y)$. The conversely holds if X is locally compact Hausdorff.*

Proof. Pg. 287 of Munkres. □

Theorem 111 (Ascoli's Theorem)**.** *Let X be a space and (Y, d) be a metric space. Give $\mathscr{C}(X, Y)$ in the topology of compact convergence; let $\mathcal{F}$ be a subset of $\mathscr{C}(X, Y)$.*

1. *If $\mathcal{F}$ is equicontinuous under d and the set*

$$\mathcal{F}_a = \{f(a) \mid f \in \mathcal{F}\}$$

has compact closure for each $a \in X$, then $\mathcal{F}$ is contined in a compact subspace of $\mathscr{C}(X, Y)$.

2. *The converse holds if X is locally compact Hausdorff.*

Proof. Pg. 290-292 of Munkres. □

6 Topological Groups

6.1 Topological Groups

Definition 85 (Topological Group)**.** A *topological group G* is a group that is also a topological space such that the map $G \times G \to G$ given by $(x, y) \mapsto xy$ and the map $G \to G$ given by $x \mapsto x^{-1}$ are both continuous.

Proposition 112 (Open Subgroups are Closed)**.** *If G is a topological group, then every open subgroup of G is also closed (and vice versa).*

Proof. Let H be an open subgroup of G. Then any coset xH is also open since the map $G \to G$ given by $g \mapsto xg$ is a homeomorphism (it is continuous by the axiom requiring the map $(x, y) \mapsto xy$ to be continuous, and invertible). Hence $Y = \bigcup_{x \in G \setminus H} xH$ is open. But notice by elementary group theory that $H = G \setminus Y$, so H is closed. Similar reasoning holds if H is a closed subgroup of G. □

Proposition 113. *If G is a topological group, and H is a subgroup of G, then the topological closure of H, $\overline{H}$, is a subgroup of G.*

Proof. Let $g, h \in \overline{H}$ and U be a neighborhood of gh. Now, recall that the map $\mu : G \times G \to G$ given by $(x, y) \mapsto xy$ is continuous, so $\mu^{-1}(U)$ is a neighborhood of (g, h). Hence there are open neigbhorhoods V_1 of g and V_2 of h such that $V_1 \times V_2 \subseteq \mu^{-1}(U)$. By Theorem 4, there exists $x \in V_1 \cap H$ and $y \in V_2 \cap H$. Now, $x, y \in H$ implies that $xy \in H$. Yet, at the same time, $(x, y) \in \mu^{-1}(U)$, so $xy \in U$. Thus $xy \in U \cap H \neq \varnothing$, and since U was an arbitrary neighborhood of gh, we have by Theorem 4 that $gh \in \overline{H}$.

Now let $\iota : G \to G$ be the inverse map $x \mapsto x^{-1}$. Let W be an open neigborhood of g^{-1}. Then $\iota^{-1}(W)$ is open and contains g, so there is a point $z \in H \cap \iota^{-1}(W)$ by Theorem 4. Yet then $z \in H$ implies $z^{-1} \in H$, and $z \in \iota^{-1}(W)$ implies that $z^{-1} \in W$, so $z^{-1} \in H \cap W$. Since W was again an arbitrary neighborhood of g^{-1}, we have by Theorem 4 that $g^{-1} \in \overline{H}$. □

6.2 Separation and Countability Axioms

We will need a lemma for the following proof:

Lemma 114 (Criterion for Hausdorff Space)**.** *A topological space X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) \mid x \in X\}$ is closed in $X \times X$.*

Proof. Suppose Δ is closed in $X \times X$. Take distinct points x and y and consider the point $\langle x, y \rangle$. Since x and y are distinct, $\langle x, y \rangle$ is not in Δ. Therefore, $\langle x, y \rangle$ belongs to the open set $(X \times X) \setminus D$. By definition of the product topology on $X \times X$, this implies that $\langle x, y \rangle$ belongs to an open set of the form $U \times V$ contained in $(X \times X) \setminus D$, where U and V are open in X. Yet then U and V are neighborhoods of x and y, and they are disjoint since they have empty intersection with the diagonal.

Now suppose X is Hausdorff. Take any point $\langle x, y\rangle \in (X \times X) \setminus D$. Now, x and y are contained in disjoint neighborhoods U_x and U_y. Their product $U_x \times U_y$ is open and has empty intersection with D. Yet then

$$(X \times X) \setminus D = \bigcup_{x \neq y} (U_x \times U_y)$$

is the union of open subsets, so it is open. Hence D is closed, as desired. □

Proposition 115. *Any topological group which is T_1 is Hausdorff.*

Proof. Consider the map $\phi : G \times G \to G$ given by $(x, y) \mapsto xy^{-1}$. ϕ is continuous by the axioms of a topological group and the fact that the composition of continuous maps is continuous. If G is T_1, then $\{1\}$ is closed, so $\phi^{-1}(1) = \Delta$ is closed by Theorem 5. Then by Lemma 114, we are done. □

Theorem 116. *Every topological group is regular.*

Proof. Consider the map $f : G \times G \to G$ defined by $f(a, b) = ab^{-1}$. This map is continuous by the axioms of a topological group. Now take some point x and a closed set B of G not containing x. Define $U = G \setminus B$. To prove that G is regular, it suffices to find open sets V and W such that $x \in V$, $X \setminus U \subseteq W$, and $V \cap W = \varnothing$.

Now, if $x \in U$, then $f^{-1}(U)$ contains (x, e). Hence by the definition of the product topology $(x, e) \in V \times W' \subseteq f^{-1}(U)$ for some open subsets V and W' such that $x \in V$ and $e \in W'$. Now define $W = (X \setminus U)W'$. $W \supseteq (X \setminus U)$, since W' contains e by construction.

Now, of the sake of contradiction, assume $V \cap W$ is nonempty. Let $y \in V \cap W$. Then, since $W = (X \setminus U)W'$, $y = cd$ for $c \notin U$ and $d \in W'$. Hence $(y, d) \in V \times W \subseteq f^{-1}(U)$, so $yd^{-1} = c \in U$, a contradiction. □

Theorem 117. *Let G be a locally compact connected T_1 topological group. Then G is paracompact.*

Proof. Now, G is regular, so it suffices to show that G is Lindelöf by Theorem 90. For this, it suffices further to show that G is σ-compact by Theorem 56. Now, by the local compactness criterion, there is a compact subspace C of X that contains e. Since G is Hausdorff by Proposition 115, C is closed.

Let U' be the interior of X, so $\overline{U'} = C$ is compact. Let ι be the inversion map $G \to G$ given by $g \mapsto g^{-1}$. Define $U'^{-1} = \iota(U')$. Since ι is a homeomorphism, $\iota(\overline{U'}) = \overline{\iota(U')}$. Hence $C^{-1} = \iota(C)$ is a compact subspace containing U'^{-1}. Therefore, $U = U' \cup U'^{-1}$ satisfies $\overline{U} = \overline{U'} \cup \overline{U'^{-1}} = C \cup C^{-1}$, which is compact since it is the union of two compact subspaces. Furthermore, U is symmetric about e.

Now, I claim that U^n is open for any $n \in \mathbb{Z}^+$ by induction. Clearly, the base case $n = 1$ is complete. Therefore, assume U^n is open. Then, since multiplication by u is a homeomorphism, uU^n is open. But then $U^{n+1} = \bigcup_{u \in U} uU^n$ is the union of open spaces and therefore open, as desired. By similar logic, $\overline{U^n}$ has compact closure for any $n \in \mathbb{Z}^+$.

Yet it is easy to verify that $H = \bigcup_{n \in \mathbb{Z}^+} U^n$ is a subgroup, and indeed an open subgroup. By Proposition 112, H is also closed, so by the connectedness of G we must have $H = G$. Yet $G = \bigcup_{n \in \mathbb{Z}^+} \overline{U^n}$, so G is the union of countably many compact subspaces and therefore is compact itself. □